

Weak and strong no-arbitrage conditions for continuous financial markets

Claudio Fontana

INRIA Paris - Rocquencourt,
Domaine de Voluceau, Rocquencourt, BP 105, Le Chesnay Cedex, 78153, France
and

Laboratoire Analyse et Probabilités, Université d'Evry Val d'Essonne
23 Boulevard de France, Evry Cedex, 91037, France.

E-mail: `claudio.fontana@inria.fr`

This version: February 27, 2013

Abstract

We propose a unified analysis of a whole spectrum of no-arbitrage conditions for financial market models based on continuous semimartingales. In particular, we focus on no-arbitrage conditions weaker than the classical notions of *No Arbitrage* and *No Free Lunch with Vanishing Risk*. We provide a complete characterisation of all no-arbitrage conditions, linking their validity to the existence and to the properties of *(weak) martingale deflators* and to the characteristics of the discounted asset price process.

Keywords: arbitrage, continuous semimartingale, market price of risk, martingale deflator, arbitrage of the first kind, free lunch with vanishing risk.

Mathematics Subject Classification (2010): 60G44, 60H05, 91B70, 91G10.

1 Introduction

Modern mathematical finance is strongly rooted on the *no-arbitrage* paradigm. In a nutshell, this amounts to excluding the possibility of “making money out of nothing” by cleverly trading in the financial market. Since the existence of arbitrage possibilities is both unrealistic and conflicts with the existence of an equilibrium, any mathematical model which aims at representing the functioning of a realistic financial market is required to satisfy a suitable no-arbitrage condition, in the absence of which one cannot draw meaningful conclusions on asset prices and investors’ behavior.

The search for a satisfactory no-arbitrage condition has a rather long history, grown at the border between financial economics and mathematics. We do not attempt here a detailed overview of the historical developments of the modern no-arbitrage theory, but only mention the seminal papers [26]-[27] and refer the reader to [17] and [60] for more information. A decisive step in this

history was marked by the paper [11], where, in the case of locally bounded semimartingales, the authors proved the equivalence between the *No Free Lunch with Vanishing Risk (NFLVR)* condition (a condition slightly stronger than the classical *No Arbitrage (NA)* condition) and the existence of an *Equivalent Local Martingale Measure (ELMM)*, i.e., a new probability measure equivalent to the original one such that the discounted asset price process is a local martingale under the new measure. The result was then extended to general semimartingale models in the subsequent papers [16] and [35].

The NFLVR condition has established itself as a golden standard and the vast majority of models proposed in quantitative finance satisfies it. However, financial market models which fail to satisfy the NFLVR condition have also appeared in recent years. In particular, in the context of *Stochastic Portfolio Theory* (see [21]-[22]), the NFLVR condition is not imposed as a normative assumption and it is shown that arbitrage opportunities may naturally arise in the market. A similar perspective is also taken in the *Benchmark Approach* (see [51]-[54]), the main goal of which consists in the development of an asset pricing theory which does not rely on the existence of ELMMs. Related works that explicitly consider situations where NFLVR may fail are [5], [10], [30], [38]-[40] and also, in the more specific case of diffusion models, [24], [43], [44] and [59]. Somewhat surprisingly, these works have demonstrated that the full strength of NFLVR is not necessarily needed in order to solve the fundamental problems of valuation, hedging and portfolio optimisation. However, the situation is made more complicated by the fact that many different and alternative no-arbitrage conditions have been proposed in the literature during the last two decades.

Motivated by the preceding discussion, the present paper aims at presenting a unified and clear perspective on the most significant no-arbitrage conditions in the context of general financial market models based on continuous semimartingales¹. In particular, we carefully study several no-arbitrage conditions which are *weaker* than the classical and *strong* NA and NFLVR conditions, namely the *No Increasing Profit (NIP)*, *No Strong Arbitrage (NSA)* and *No Arbitrage of the First Kind (NA1)* conditions. We prove the following implications:

$$\text{NFLVR} \implies \text{NA1} \implies \text{NSA} \implies \text{NIP}. \quad (1.1)$$

By means of explicit examples and counterexamples, we illustrate these implications and discuss their economic meaning as well as the connections to several other conditions which have appeared in the literature, thus providing a complete picture of the whole spectrum of no-arbitrage conditions. Moreover, we prove that none of the converse implications in (1.1) holds in general.

We show that *weak* no-arbitrage conditions (NIP, NSA and NA1) can be fully characterised in terms of the semimartingale characteristics of the discounted price process, while this is in general impossible for the *strong* no-arbitrage conditions (NA and NFLVR), since the latter also depend on the structure of the underlying filtration. Moreover, we link the validity of different no-arbitrage conditions to the existence and the properties of *weak martingale deflators*, weaker counterparts of density processes of ELMMs. In particular, we show that the weak NSA and NA1 conditions (as well as their equivalent formulations) can be directly checked by looking at the properties of a single *minimal* weak martingale deflator, whose behavior is determined by the *mean-variance trade-off* process associated with the discounted price process. Furthermore, we prove that the NA1

¹The continuous semimartingale setting covers many models widely used in quantitative finance and, in particular, almost all models developed in the context of Stochastic Portfolio Theory.

condition (as well as its equivalent formulations) is stable with respect to changes of numéraire (see Corollary 5.6), unlike the classical NFLVR condition, and allows to recover NFLVR by means of a suitable change of numéraire (see Corollary 6.7).

To the best of our knowledge, there does not exist in the literature a similar unifying analysis of the weak no-arbitrage conditions going beyond the classical notions of NA and NFLVR. The only partial exception is contained in Chapter 1 of [29]. In comparison with the latter work, our approach puts an emphasis on the role of weak martingale deflators and also carefully takes into account minimal no-arbitrage conditions which are weaker than the NUPBR condition, on which the presentation in [29] is focused. Moreover, besides providing different and original proofs, we explicitly study the NIP, NA1, NCT and NAA no-arbitrage conditions (see e.g. the table in Section 7) which are not discussed in [29] and drop the non-negativity assumption on the discounted asset prices.

The paper is structured as follows. Section 2 presents the general setting and introduces the main no-arbitrage conditions which shall be studied in the following. Section 3, 4 and 5 discuss the NIP, NSA and NA1 conditions, respectively. Section 6 deals with the classical NA and NFLVR conditions and discusses the relations with the previous no-arbitrage conditions. Finally, Section 7 concludes by summarising the main results of the paper.

2 General setting and preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a given filtered probability space, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ is assumed to satisfy the usual conditions of right-continuity and P -completeness and, for the sake of simplicity, $T \in (0, \infty)$ represents a finite time horizon (all the results we are going to present can be rather easily adapted to the infinite horizon case). Note that the initial σ -field \mathcal{F}_0 is not assumed to be trivial. Let \mathcal{M} be the family of all uniformly integrable P -martingales and \mathcal{M}_{loc} the family of all local P -martingales. Without loss of generality, we assume that all processes in \mathcal{M}_{loc} have càdlàg paths and we denote by \mathcal{M}^c and $\mathcal{M}_{\text{loc}}^c$ the families of all processes in \mathcal{M} and \mathcal{M}_{loc} , respectively, with continuous paths.

We consider a financial market comprising $d + 1$ assets, whose prices are represented by the \mathbb{R}^{d+1} -valued process $\tilde{S} = (\tilde{S}_t)_{0 \leq t \leq T}$, with $\tilde{S}_t = (\tilde{S}_t^0, \tilde{S}_t^1, \dots, \tilde{S}_t^d)^\top$, with $^\top$ denoting transposition. We assume that \tilde{S}_t^0 is P -a.s. strictly positive for all $t \in [0, T]$ and, as usual in the literature, we then take asset 0 as *numéraire* and express all quantities in terms of \tilde{S}^0 . This means that the $(\tilde{S}^0$ -discounted) price of asset 0 is constant and equal to 1 and the remaining d risky assets have $(\tilde{S}^0$ -discounted) prices described by the \mathbb{R}^d -valued process $S = (S_t)_{0 \leq t \leq T}$, where $S_t^i := \tilde{S}_t^i / \tilde{S}_t^0$ for all $t \in [0, T]$ and $i = 1, \dots, d$. The process S is assumed to be a *continuous* \mathbb{R}^d -valued semimartingale on $(\Omega, \mathcal{F}, \mathbb{F}, P)$. In particular, S is a special semimartingale, admitting the unique canonical decomposition $S = S_0 + A + M$, where A is a continuous \mathbb{R}^d -valued predictable process of finite variation and M is an \mathbb{R}^d -valued process in $\mathcal{M}_{\text{loc}}^c$ with $M_0 = A_0 = 0$. Due to Proposition II.2.9 of [32], it holds that, for all $i, j = 1, \dots, d$,

$$A^i = \int a^i dB \quad \text{and} \quad \langle S^i, S^j \rangle = \langle M^i, M^j \rangle = \int c^{ij} dB \quad (2.1)$$

where B is a continuous real-valued predictable increasing process, $a = (a^1, \dots, a^d)^\top$ is an \mathbb{R}^d -valued predictable process and $c = ((c^{i1})_{1 \leq i \leq d}, \dots, (c^{id})_{1 \leq i \leq d})$ is a predictable process taking values in the

cone of symmetric nonnegative $d \times d$ matrices. The processes a , c and B satisfying (2.1) are not unique in general, but our results do not depend on the specific choice we make (for instance, B can be taken as $B = \sum_{i=1}^d \text{Var}(A^i) + \sum_{i=1}^d \langle M^i \rangle$, with $\text{Var}(\cdot)$ denoting the total variation). Note that we do not necessarily assume that S takes values in the positive orthant of \mathbb{R}^d . For every $t \in [0, T]$, let us denote by c_t^+ the Moore-Penrose pseudoinverse of the matrix c_t . The proof of Proposition 2.1 of [19] shows that the process $c^+ = (c_t^+)_{0 \leq t \leq T}$ is predictable and, hence, the process a can be represented as:

$$a = c \lambda + \nu \quad (2.2)$$

where $\lambda = (\lambda_t)_{0 \leq t \leq T}$ is the \mathbb{R}^d -valued predictable process defined by $\lambda_t := c_t^+ a_t$, for all $t \in [0, T]$, and $\nu = (\nu_t)_{0 \leq t \leq T}$ is an \mathbb{R}^d -valued predictable process with $\nu_t \in \text{Ker}(c_t) := \{x \in \mathbb{R}^d : c_t x = 0\}$ for all $t \in [0, T]$.

We suppose that the financial market is frictionless, meaning that there are no trading restrictions, transaction costs, liquidity effects or other market imperfections. In order to mathematically describe the activity of trading, we need to introduce the notion of *admissible strategy*. To this effect, let $L(S)$ be the set of all \mathbb{R}^d -valued S -integrable predictable processes, in the sense of [32], and, for $H \in L(S)$, denote by $H \cdot S$ the stochastic integral process $(\int_0^t H_u dS_u)_{0 \leq t \leq T}$. Since S is a continuous semimartingale, Proposition III.6.22 of [32] implies that $L(S) = L_{\text{loc}}^2(M) \cap L^0(A)$, where $L_{\text{loc}}^2(M)$ and $L^0(A)$ are the sets of all \mathbb{R}^d -valued predictable processes H such that $\int_0^T H_t^\top d\langle M, M \rangle_t H_t < \infty$ P -a.s. and $\int_0^T |H_t^\top dA_t| < \infty$ P -a.s., respectively. Hence, due to (2.1), an \mathbb{R}^d -valued predictable process H belongs to $L(S)$ if and only if

$$\int_0^T v(H)_t dB_t < \infty \text{ } P\text{-a.s.} \quad \text{where} \quad v(H)_t := \sum_{i,j=1}^d H_t^i c_t^{ij} H_t^j + \left| \sum_{i=1}^d H_t^i a_t^i \right|.$$

Remark 2.1. The set $L(S)$ represents the most general class of predictable integrands with respect to S . In particular, it contains non-locally bounded integrands, as in [7]. Note that, for $H \in L(S)$, we have $H \cdot M \in \mathcal{M}_{\text{loc}}^c$ and the continuous semimartingale $H \cdot S$ admits the unique canonical decomposition $H \cdot S = H_0 S_0 + H \cdot A + H \cdot M$. We also want to emphasize that $H \cdot S$ has to be understood as the vector stochastic integral of H with respect to S and is in general different from the sum of the componentwise stochastic integrals $\sum_{i=1}^d \int H^i dS^i$; see for instance [31] and [63].

We are now in a position to formulate the following classical definition.

Definition 2.2. Let $a \in \mathbb{R}_+$. An element $H \in L(S)$ is said to be an a -admissible strategy if $H_0 = 0$ and $(H \cdot S)_t \geq -a$ P -a.s. for all $t \in [0, T]$. An element $H \in L(S)$ is said to be an admissible strategy if it is an a -admissible strategy for some $a \in \mathbb{R}_+$.

For $a \in \mathbb{R}_+$, we denote by \mathcal{A}_a the set of all a -admissible strategies and by \mathcal{A} the set of all admissible strategies, i.e., $\mathcal{A} = \bigcup_{a \in \mathbb{R}_+} \mathcal{A}_a$. As usual, H_t^i represents the number of units of asset i held in the portfolio at time t . The condition $H_0 = 0$ amounts to requiring that the initial position in the risky assets is zero and, hence, the initial endowment is entirely expressed in terms of the numéraire asset. For $H \in \mathcal{A}$, we define the *gains from trading* process $G(H) = (G_t(H))_{0 \leq t \leq T}$ by $G_t(H) := (H \cdot S)_t$, for all $t \in [0, T]$. According to Definition 2.2, the process $G(H)$ associated to an admissible strategy $H \in \mathcal{A}$ is uniformly bounded from below by some constant. This restriction is needed since the set $L(S)$ is too large for the purpose of modeling reasonable trading strategies and may also contain doubling strategies. This possibility is automatically ruled out if we impose a

limit to the line of credit which can be granted to every market participant. For $(x, H) \in \mathbb{R}_+ \times \mathcal{A}$, we define the *portfolio value* process $V(x, H) = (V_t(x, H))_{0 \leq t \leq T}$ by $V(x, H) := x + G(H)$. This corresponds to consider portfolios which are generated by *self-financing* admissible strategies.

We now introduce five main notions of arbitrage which shall be studied in this paper.

Definition 2.3.

- (i) An element $H \in \mathcal{A}_0$ generates an increasing profit if the process $G(H)$ is predictable² and if $P(G_s(H) \leq G_t(H), \text{ for all } 0 \leq s \leq t \leq T) = 1$ and $P(G_T(H) > 0) > 0$. If there exists no such $H \in \mathcal{A}_0$ we say that the No Increasing Profit (NIP) condition holds;
- (ii) an element $H \in \mathcal{A}_0$ generates a strong arbitrage opportunity if $P(G_T(H) > 0) > 0$. If there exists no such $H \in \mathcal{A}_0$, i.e., if $\{G_T(H) : H \in \mathcal{A}_0\} \cap L_+^0 = \{0\}$, we say that the No Strong Arbitrage (NSA) condition holds;
- (iii) a non-negative random variable ξ generates an arbitrage of the first kind if $P(\xi > 0) > 0$ and for every $v \in (0, \infty)$ there exists an element $H^v \in \mathcal{A}_v$ such that $V_T(v, H^v) \geq \xi$ P -a.s. If there exists no such random variable ξ we say that the No Arbitrage of the First Kind (NA1) condition holds;
- (iv) an element $H \in \mathcal{A}$ generates an arbitrage opportunity if $G_T(H) \geq 0$ P -a.s. and $P(G_T(H) > 0) > 0$. If there exists no such $H \in \mathcal{A}$, i.e., if $\{G_T(H) : H \in \mathcal{A}\} \cap L_+^0 = \{0\}$, we say that the No Arbitrage (NA) condition holds;
- (v) a sequence $\{H^n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ generates a Free Lunch with Vanishing Risk if there exist an $\varepsilon > 0$ and an increasing sequence $\{\delta_n\}_{n \in \mathbb{N}}$ with $0 \leq \delta_n \nearrow 1$ such that $P(G_T(H^n) > -1 + \delta_n) = 1$ and $P(G_T(H^n) > \varepsilon) \geq \varepsilon$. If there exists no such sequence we say that the No Free Lunch with Vanishing Risk (NFLVR) condition holds.

The NIP condition is similar to the No Unbounded Increasing Profit condition introduced in [38] and represents the strongest notion of arbitrage among those listed above. The adjective “unbounded” in the definition adopted in [38] can be explained as follows: if $H \in \mathcal{A}_0$ yields an increasing profit in the sense of Definition 2.3-(i), we have $H^n := nH \in \mathcal{A}_0$ and $G(H^n) \geq G(H)$, for every $n \in \mathbb{N}$. This means that the increasing profit generated by H can be scaled to arbitrarily large levels of wealth. The NSA condition corresponds to the NA^+ condition studied in [64]. The notion of arbitrage of the first kind has been introduced in [39]. The NA and NFLVR conditions are classical and, in particular, they go back to the seminal papers [26], [27] and [11]. The NA condition can be equivalently formulated as $\mathcal{C} \cap L_+^\infty = \{0\}$, where $\mathcal{C} := (\{G_T(H) : H \in \mathcal{A}\} - L_+^0) \cap L^\infty$. The above definition of NFLVR is taken from [38] and can be shown to be equivalent to $\overline{\mathcal{C}} \cap L_+^\infty = \{0\}$, with the bar denoting the closure in the norm topology of L^∞ , as in [11]. In the following sections, we shall also examine several other no-arbitrage conditions equivalent to the ones introduced in Definition 2.3.

²The reason for requiring $G(H)$ to be predictable will become clear in Theorem 3.1, which is formulated with respect to general locally square-integrable semimartingales, in the sense of Definition II.2.27 of [32]. Of course, as soon as S is continuous, we may drop the predictability requirement.

3 No Increasing Profit

An increasing profit represents an investment opportunity which does not require any initial investment nor any line of credit and, moreover, generates an increasing wealth process, yielding a non-zero final wealth with strictly positive probability. From a financial point of view, the notion of increasing profit represents the most egregious form of arbitrage and, therefore, should be banned from any reasonable financial model. The following theorem characterises the NIP condition. At no extra cost, we state and prove the result for general locally square-integrable semimartingales, in the sense of Def. II.2.27 of [32].

Theorem 3.1. *The following are equivalent, using the notation introduced in (2.1)-(2.2):*

- (i) *the NIP condition holds;*
- (ii) *for every $H \in L(S)$, $H_t^\top c_t = 0$ $P \otimes B$ -a.e. implies $H_t^\top a_t = 0$ $P \otimes B$ -a.e.;*
- (iii) *$\nu_t = 0$ $P \otimes B$ -a.e.*

Proof. (i) \Rightarrow (ii): Let us define the product space $\overline{\Omega} := \Omega \times [0, T]$. Suppose that NIP holds and let $H = (H_t)_{0 \leq t \leq T}$ be a process in $L(S)$ such that $H_t^\top c_t = 0$ $P \otimes B$ -a.e. (so that $H \cdot M = 0$) but $P \otimes B((\omega, t) \in \overline{\Omega} : H_t^\top(\omega) a_t(\omega) \neq 0) > 0$. By the Hahn-Jordan decomposition (see [13], Theorem 2.1), we can write $H \cdot A = \int (\mathbf{1}_{D^+} - \mathbf{1}_{D^-}) dV$, where D^+ and D^- are two disjoint predictable subsets of $\overline{\Omega}$ such that $D^+ \cup D^- = \overline{\Omega}$ and $V := \text{Var}(H \cdot A)$. Let $\psi := \mathbf{1}_{D^+} - \mathbf{1}_{D^-}$ and define the \mathbb{R}^d -valued predictable process $\tilde{H} := \psi H \mathbf{1}_{(0, T]}$. Due to the associativity of the stochastic integral, it is clear that $\tilde{H} \in L(S)$ and $\tilde{H} \cdot M = 0$. Thus, using again the associativity of the stochastic integral:

$$\tilde{H} \cdot S = \tilde{H} \cdot A = (\psi H) \cdot A = \psi \cdot (H \cdot A) = \psi^2 \cdot V = V.$$

The process V is non-negative, increasing and predictable and satisfies $P(V_T > 0) > 0$, since $H \cdot A$ is supposed to be not identically zero. Clearly, this amounts to saying that \tilde{H} generates an increasing profit, thus contradicting the assumption that NIP holds. Hence, it must be $H_t^\top a_t = 0$ $P \otimes B$ -a.e.

(ii) \Rightarrow (iii): let $H = (H_t)_{0 \leq t \leq T}$ be an \mathbb{R}^d -valued predictable process such that $\|H_t(\omega)\| \in \{0, 1\}$ for all $(\omega, t) \in \overline{\Omega}$. Since $H_t^\top c_t = 0$ $P \otimes B$ -a.e. implies that $H_t^\top a_t = 0$ $P \otimes B$ -a.e., condition (iii) follows directly from the absolute continuity result of Theorem 2.3 of [13].

(iii) \Rightarrow (i): suppose that $\nu_t = 0$ $P \otimes B$ -a.e. and let $H \in \mathcal{A}_0$ generate an increasing profit. The process $G(H) = H \cdot S$ is increasing, hence of finite variation, and predictable. In particular, $H \cdot S$ is a special semimartingale and, hence, due to Proposition 2 of [31], we can write $H \cdot S = H \cdot A + H \cdot M$. This implies that $H \cdot M = H \cdot S - H \cdot A$ is also predictable and of finite variation. Theorem III.15 of [55] then implies that $H \cdot M = 0$, being a predictable local martingale of finite variation. Hence, due to the Kunita-Watanabe inequality, for all $t \in [0, T]$ and $i = 1, \dots, d$:

$$|\langle H \cdot M, M^i \rangle_t| \leq \sqrt{\langle H \cdot M \rangle_t} \sqrt{\langle M^i \rangle_t} = 0 \quad P\text{-a.s.}$$

since $\langle M^i \rangle_t < \infty$ P -a.s. for all $t \in [0, T]$ and $i = 1, \dots, d$. Hence, for all $t \in [0, T]$:

$$G_t(H) = (H \cdot A)_t = \int_0^t H_u^\top a_u dB_u = \int_0^t H_u^\top c_u \lambda_u dB_u = \int_0^t d\langle H \cdot M, M \rangle_u \lambda_u = 0 \quad P\text{-a.s.}$$

In particular $P(G_T(H) > 0) = 0$, thus contradicting the hypothesis that H generates an increasing profit. \square

Clearly, $\nu_t = 0$ $P \otimes B$ -a.e. means that $dA \ll d\langle M, M \rangle$. The latter condition is known in the literature as the *weak structure condition* and the process λ is then usually referred to as the *instantaneous market price of risk* (see e.g. [30], Section 3). We want to point out that results similar to Theorem 3.1 have already appeared in the literature, albeit under stronger assumptions. In particular, Theorem 3.5 of [13] (see also [56], Theorem 1) shows that $dA \ll d\langle M, M \rangle$ holds under the classical NA condition, which is strictly stronger than NIP (see Section 6). Somewhat more generally, [36] (see also [29], Theorem 1.13) prove that $dA \ll d\langle M, M \rangle$ holds under the NSA condition, which is also strictly stronger than the NIP condition (see Section 4). Theorem 3.1 shows that the weak structure condition $dA \ll d\langle M, M \rangle$ is *equivalent* to the NIP condition, which represents an indispensable no-arbitrage requirement for any reasonable financial market.

Remarks 3.2. 1) As can be seen by inspecting the proof of Theorem 3.1, the NIP condition is also equivalent to the absence of elements $H \in \mathcal{A}_0$ such that the gains from trading process $G(H)$ is predictable and of finite variation (not necessarily increasing) and satisfies $P(G_T(H) > 0) > 0$.

2) In general, as soon as the NIP condition holds, there may exist many \mathbb{R}^d -valued predictable processes $\gamma = (\gamma_t)_{0 \leq t \leq T}$ such that $dA_t = d\langle M, M \rangle_t \gamma_t$. However, for any such process γ , elementary linear algebra gives $\Pi_{c_t}(\gamma_t) = \lambda_t$, where we denote by $\Pi_{c_t}(\cdot)$ the orthogonal projection onto the range of the matrix c_t , for all $t \in [0, T]$. In turn, this implies that $\int_0^T \gamma_t^\top c_t \gamma_t dB_t \geq \int_0^T \lambda_t^\top c_t \lambda_t dB_t$ for every $t \in [0, T]$, thus showing a *minimality* property of the process λ introduced in (2.2).

Example 3.3. We now give an explicit example of a model allowing for increasing profits, using the concept of local time of a continuous local martingale. Let $N = (N_t)_{0 \leq t \leq T} \in \mathcal{M}_{\text{loc}}^c$ and let $S := |N|$. Tanaka's formula (see [57], Theorem VI.1.2) gives the following canonical decomposition:

$$S_t = |N_0| + \int_0^t \text{sign}(N_u) dN_u + L_t^0, \quad \text{for all } t \in [0, T],$$

where the process $L^0 = (L_t^0)_{0 \leq t \leq T}$ is the local time at the level 0 of N . Using the notation introduced in Section 2, we have $A = L^0$ and $M = \text{sign}(N) \cdot N$. We now show that $dA \ll d\langle M, M \rangle$ does not hold, where $\langle M, M \rangle = \int (\text{sign}(N))^2 d\langle N \rangle = \langle N \rangle$. In fact, Proposition VI.1.3 of [57] shows that, for almost all $\omega \in \Omega$, the measure (in t) $dL_t^0(\omega)$ is carried by the set $\{t : N_t(\omega) = 0\}$. However, for almost all $\omega \in \Omega$, the set $\{t : N_t(\omega) = 0\}$ has zero measure with respect to $d\langle N \rangle_t(\omega)$. This means that L^0 induces a measure which is singular with respect to the measure induced by $\langle N \rangle$. Theorem 3.1 then implies that NIP fails.

In the context of this example, we can also explicitly construct a trading strategy which yields an increasing profit. For simplicity, let us suppose that $N_0 = 0$ P -a.s. and define the process $H = (H_t)_{0 \leq t \leq T}$ by $H := \mathbf{1}_{\{N=0\} \cap [0, T]}$. Clearly, H is a bounded predictable process and so $H \in L(S)$. Furthermore, we have $(H \cdot M)_t = \int_0^t H_u \text{sign}(N_u) dN_u = 0$ P -a.s. for all $t \in [0, T]$, due to the fact that

$$\left\langle \int H \text{sign}(N) dN \right\rangle_T = \int_0^T H_u^2 (\text{sign}(N_u))^2 d\langle N \rangle_u = \int_{0+}^T \mathbf{1}_{\{N_u=0\}} d\langle N \rangle_u = 0 \quad P\text{-a.s.}$$

Note also that $\int H dL^0 = L^0$, since $dL_t^0(\omega)$ is carried by the set $\{t : N_t(\omega) = 0\}$ for almost all $\omega \in \Omega$. Hence, for all $t \in [0, T]$:

$$(H \cdot S)_t = \int_0^t H_u \text{sign}(N_u) dN_u + \int_0^t H_u dL_u^0 = L_t^0 = \sup_{s \leq t} \left(- \int_0^s \text{sign}(N_u) dN_u \right)$$

where the last equality follows from Skorohod's lemma (see [57], Lemma VI.2.1). This shows that the gains from trading process $G(H) = H \cdot S$ starts from 0 and is non-decreasing. In particular, this implies $H \in \mathcal{A}_0$. Finally, if we assume that the local martingale N is not identically zero, we also have $P(G_T(H) > 0) > 0$. Indeed, suppose on the contrary that $P(G_T(H) > 0) = 0$, so that $\sup_{s \leq T} (-\int_0^s \text{sign}(N_u) dN_u) = 0$ P -a.s. and, hence, $\int_0^t \text{sign}(N_u) dN_u \geq 0$ P -a.s. for all $t \in [0, T]$. Due to Fatou's lemma, this implies that the process $\text{sign}(N) \cdot N$ is a non-negative supermartingale, being a non-negative continuous local martingale. Since $(\text{sign}(N) \cdot N)_0 = 0$, the supermartingale property gives $\text{sign}(N) \cdot N = 0$, which in turn implies that $\langle N \rangle = \langle \text{sign}(N) \cdot N \rangle = 0$, thus contradicting the assumption that N is not trivial.

Remark 3.4. An interesting interpretation of the arbitrage possibilities arising from local times can be found in [33], where it is shown that the existence of large traders (whose orders affect market prices) can introduce “hidden” arbitrage opportunities for the small traders, who act as price-takers. These arbitrage profits are “hidden” since they occur on time sets of Lebesgue measure zero, being related to the local time of Brownian motion. Another example of a financial market which allows for arbitrage profits arising from local time can be found in the recent paper [50].

4 No Strong Arbitrage

A strong arbitrage opportunity consists of an investment opportunity which does not require any initial capital nor any line of credit and leads to a non-zero final wealth with strictly positive probability. Of course, this sort of strategy should be banned from any reasonable financial market, since every agent would benefit in an unlimited way from a strong arbitrage opportunity. According to Definition 2.3, it is evident that an increasing profit generates a strong arbitrage opportunity. Two examples of models which satisfy NIP but which allow for strong arbitrage opportunities will be presented at the end of this section, thus showing that NSA is strictly stronger than NIP. The notion of strong arbitrage opportunity is similar to the concept of *arbitrage opportunity* adopted in Section 3 of [43]. Furthermore, it also corresponds to the notion of arbitrage adopted in the context of the *Benchmark Approach*³, see e.g. Section 7 of [53] and Section 10.3 of [54]. We also want to mention that the NSA condition has been also assumed in [10] as a necessary requirement in order to construct the *Growth Optimal Portfolio (GOP)*.

Let us now introduce another notion of arbitrage, which has been originally formulated in [13] and will turn out to be equivalent to the notion of strong arbitrage opportunity introduced in Definition 2.3-(ii).

Definition 4.1. An element $H \in \mathcal{A}_0$ generates an immediate arbitrage opportunity if there exists a stopping time τ such that $P(\tau < T) > 0$ and if $H = H\mathbf{1}_{(\tau, T]}$ and $G_t(H) > 0$ P -a.s. for all $t \in (\tau, T]$. If there exists no such $H \in \mathcal{A}_0$ we say that the No Immediate Arbitrage (NIA) condition holds.

We have the following simple lemma (compare also with [13], Lemma 3.1).

³We want to make the reader aware of the fact that typical applications of the Benchmark Approach require assumptions stronger than the NSA condition, namely the existence of the GOP. In Theorem 4.12 of [38], it is shown that the existence of a (non-exploding) GOP is equivalent to the *No Unbounded Profit with Bounded Risk (NUPBR)* condition, which is strictly stronger than NSA (as we are going to show in Section 5). This means that, in the context of the Benchmark Approach, not only strong arbitrage opportunities but also weaker forms of arbitrage must be ruled out from the market.

Lemma 4.2. *The NSA condition and the NIA condition are equivalent.*

Proof. Suppose that $H \in \mathcal{A}_0$ generates a strong arbitrage opportunity and define the stopping time $\tau := \inf\{t \in [0, T] : G_t(H) > 0\} \wedge T$. Since $P(G_T(H) > 0) > 0$, we have $P(\tau < T) > 0$. For a sequence $\{\theta_n\}_{n \in \mathbb{N}}$ dense in $(0, 1)$, let us define the process $\tilde{H} := \sum_{n=1}^{\infty} 2^{-n} H \mathbf{1}_{(\tau, (\tau + \theta_n) \wedge T]}$. Clearly, we have $\tilde{H} \in \mathcal{A}_0$. Furthermore, on the event $\{\tau < T\}$ it holds that, for every $\varepsilon > 0$:

$$G_{\tau+\varepsilon}(\tilde{H}) = (\tilde{H} \cdot S)_{\tau+\varepsilon} = \sum_{n=1}^{\infty} \frac{1}{2^n} ((H \cdot S)_{(\tau+(\varepsilon \wedge \theta_n)) \wedge T} - (H \cdot S)_{\tau}) = \sum_{n=1}^{\infty} \frac{1}{2^n} (H \cdot S)_{(\tau+(\varepsilon \wedge \theta_n)) \wedge T} > 0 \text{ } P\text{-a.s.}$$

thus showing that \tilde{H} generates an immediate arbitrage opportunity at the stopping time τ . Conversely, it can be easily seen directly from Definitions 2.3-(ii) and 4.1 that an immediate arbitrage opportunity is also a strong arbitrage opportunity. \square

Recall that, due to Theorem 3.1, the NIP condition is equivalent to the validity of the relation $a = c \lambda \text{ } P \otimes B\text{-a.e.}$, where the processes a , c , λ and B are as in (2.1)-(2.2). Since NSA (or, equivalently, NIA) is stronger than NIP, it is natural to expect that NSA will imply some additional properties of the process λ . This is confirmed by the next theorem. As a preliminary, let us define the *mean-variance trade-off process* $\hat{K} = (\hat{K}_t)_{0 \leq t \leq T}$ as:

$$\hat{K}_t := \int_0^t \lambda'_u d\langle M, M \rangle_u \lambda_u = \int_0^t \lambda_u^\top c_u \lambda_u dB_u = \int_0^t a_u^\top c_u^+ a_u dB_u, \quad \text{for all } t \in [0, T]. \quad (4.1)$$

Let also $\hat{K}_s^t := \hat{K}_t - \hat{K}_s$, for all $s, t \in [0, T]$ with $s \leq t$. Following [42] and [64], we also define the stopping time

$$\alpha := \inf\{t \in [0, T] : \hat{K}_t^{t+h} = \infty, \forall h \in (0, T - t]\}$$

with the usual convention $\inf \emptyset = \infty$. The next theorem is essentially due to [64], but we opt for a slightly different proof.

Theorem 4.3. *The NSA condition holds if and only if $\nu_t = 0 \text{ } P \otimes B\text{-a.e.}$ and $\alpha = \infty \text{ } P\text{-a.s.}$*

Proof. Suppose first that NSA holds. Since NSA implies NIP, Theorem 3.1 gives that $\nu_t = 0 \text{ } P \otimes B\text{-a.e.}$ The fact that $\alpha = \infty \text{ } P\text{-a.s.}$ then follows from Theorem 3.6 of [13] together with Lemma 4.2 (compare also with [36], Sections 3-4).

Conversely, suppose that $\nu_t = 0 \text{ } P \otimes B\text{-a.e.}$ and $\alpha = \infty \text{ } P\text{-a.s.}$ and let $H \in \mathcal{A}_0$ generate a strong arbitrage opportunity. Due to Lemma 4.2, we can equivalently suppose that H generates an immediate arbitrage opportunity with respect to a stopping time τ with $P(\tau < T) > 0$. Since $P(\alpha = \infty) = 1$, we have $P(\hat{K}_\tau^{\tau+h} = \infty, \forall h \in (0, T - \tau]) = 0$. For every $n \in \mathbb{N}$, define the stopping time $\rho_n := \inf\{t > \tau : \hat{K}_\tau^t \geq n\} \wedge T$. Since \hat{K} is continuous and does not jump to infinity, it is clear that $\rho_n > \tau \text{ } P\text{-a.s.}$ on the set $\{\tau < T\}$, for all $n \in \mathbb{N}$. Let us define the predictable process $\lambda^{\tau, n} := \lambda \mathbf{1}_{(\tau, \rho_n]}$, for every $n \in \mathbb{N}$. Then, on the set $\{\tau < T\}$:

$$\int_0^T (\lambda_t^{\tau, n})^\top d\langle M, M \rangle_t \lambda_t^{\tau, n} = \int_0^T \mathbf{1}_{(\tau, \rho_n]} \lambda_t^\top d\langle M, M \rangle_t \lambda_t = \hat{K}_\tau^{\rho_n} \leq n \quad P\text{-a.s.}$$

For every $n \in \mathbb{N}$, we can define the stochastic exponential $\hat{Z}^{\tau, n} := \mathcal{E}(-\lambda^{\tau, n} \cdot M)$ as a strictly positive process in \mathcal{M}^c , due to Novikov's condition. It is obvious that $\hat{Z}^{\tau, n} = 1$ on $[0, \tau]$ and $\hat{Z}^{\tau, n} = \hat{Z}_{\rho_n}^{\tau, n}$

on $[\rho_n, T]$. We now apply the integration by parts formula to $\widehat{Z}^{\tau, n}(H \cdot S)^{\rho_n}$, where $(H \cdot S)^{\rho_n}$ denotes the process $H \cdot S$ stopped at ρ_n , and use the fact that $\widehat{Z}_t^{\tau, n} d(H \cdot A)_t^{\rho_n} = \widehat{Z}_t^{\tau, n} H_t^\top dA_t^{\rho_n}$, since $\widehat{Z}^{\tau, n} \in L(H \cdot A)$ (being $\widehat{Z}^{\tau, n}$ adapted and continuous, hence predictable and locally bounded), and the fact that $dA = d\langle M, M \rangle \lambda$ and $H = H \mathbf{1}_{(\tau, T]}$:

$$\begin{aligned} d(\widehat{Z}_t^{\tau, n}(H \cdot S)_t^{\rho_n}) &= \widehat{Z}_t^{\tau, n} d(H \cdot S)_t^{\rho_n} + (H \cdot S)_t^{\rho_n} d\widehat{Z}_t^{\tau, n} + d\langle \widehat{Z}^{\tau, n}, (H \cdot S)^{\rho_n} \rangle_t \\ &= \widehat{Z}_t^{\tau, n} d(H \cdot M)_t^{\rho_n} + \widehat{Z}_t^{\tau, n} d(H \cdot A)_t^{\rho_n} + (H \cdot S)_t^{\rho_n} d\widehat{Z}_t^{\tau, n} - \widehat{Z}_t^{\tau, n} H_t^\top d\langle M, M \rangle_t \lambda_t^{\tau, n} \\ &= \widehat{Z}_t^{\tau, n} d(H \cdot M)_t^{\rho_n} + (H \cdot S)_t^{\rho_n} d\widehat{Z}_t^{\tau, n} + \widehat{Z}_t^{\tau, n} H_t^\top (dA_t^{\rho_n} - d\langle M, M \rangle_t \lambda_t^{\tau, n}) \\ &= \widehat{Z}_t^{\tau, n} d(H \cdot M)_t^{\rho_n} + (H \cdot S)_t^{\rho_n} d\widehat{Z}_t^{\tau, n} \end{aligned}$$

This shows that $\widehat{Z}^{\tau, n}(H \cdot S)^{\rho_n}$ is a non-negative local martingale and, due to Fatou's lemma, also a supermartingale, for every $n \in \mathbb{N}$. Since $\widehat{Z}_0^{\tau, n}(H \cdot S)_0^{\rho_n} = 0$, the supermartingale property implies that $\widehat{Z}_t^{\tau, n}(H \cdot S)_t^{\rho_n} = 0$ for all $t \in [0, T]$ P -a.s., meaning that $H \cdot S = 0$ P -a.s. on $\bigcup_{n \in \mathbb{N}} [0, \rho_n]$. Since $\rho_n > \tau$ P -a.s. on $\{\tau < T\}$ and $P(\tau < T) > 0$, this contradicts the fact that $(H \cdot S)_t > 0$ P -a.s. for all $t \in (\tau, T]$, thus showing that there cannot exist an immediate arbitrage opportunity. Equivalently, due to Lemma 4.2, the NSA condition holds. \square

Theorem 4.3 shows that NSA holds as long as the mean-variance trade-off process \widehat{K} does not jump to infinity (however, \widehat{K}_T is not guaranteed to be finite). In particular, it is important to observe that we can check whether a financial market allows for strong arbitrage opportunities by looking only at the semimartingale characteristics of the discounted price process S .

We now introduce the important concept of (*weak*) *martingale deflator*, which represents a weaker counterpart to the density process of an equivalent local martingale measure (see Remark 6.4) and corresponds to the notion of *martingale density* first introduced in [61]-[62].

Definition 4.4. Let $Z = (Z_t)_{0 \leq t \leq T}$ be a non-negative local martingale with $Z_0 = 1$. We say that Z is a weak martingale deflator if the product ZS^i is a local martingale, for all $i = 1, \dots, d$. If Z satisfies in addition $Z_T > 0$ P -a.s. we say that Z is a martingale deflator.

A weak martingale deflator Z is said to be tradable if there exists a sequence $\{\theta^n\}_{n \in \mathbb{N}} \subseteq \mathcal{A}_1$ and a sequence $\{\tau_n\}_{n \in \mathbb{N}}$ of stopping times increasing P -a.s. to $\tau := \inf\{t \in [0, T] : Z_t = 0 \text{ or } Z_{t-} = 0\}$ such that $1/Z^{\tau_n} = V(1, \theta^n)$ P -a.s., for every $n \in \mathbb{N}$.

Remark 4.5. Fatou's lemma implies that any weak martingale deflator Z is a supermartingale (and also a true martingale if and only if $E[Z_T] = E[Z_0] = 1$). Furthermore, if Z is a martingale deflator, so that $Z_T > 0$ P -a.s., the minimum principle for non-negative supermartingales (see e.g. [57], Proposition II.3.4) implies that $\tau = \infty$ P -a.s. It can be verified that a martingale deflator is tradable if and only if there exists a strategy $\theta \in \mathcal{A}_1$ such that $1/Z = V(1, \theta)$ (indeed, it suffices to define $\theta := \sum_{n=1}^{\infty} \theta^n \mathbf{1}_{(\tau_{n-1}, \tau_n]}$, with $\tau_0 := 0$). This also explains the meaning of the terminology *tradable*.

We denote by $\mathcal{D}^{\text{weak}}$ and \mathcal{D} the families of all weak martingale deflators and of all martingale deflators, respectively. The next lemma shows the fundamental property of weak martingale deflators. At little extra cost, we state and prove the result for the case of general (possibly discontinuous and non-locally bounded) semimartingales (we refer to Section III.6e of [32] and to [37] for the definition and the main properties of σ -martingales). The proof is postponed to the Appendix.

Lemma 4.6. *Let $Z \in \mathcal{D}^{\text{weak}}$. Then, for any $H \in L(S)$, the product $Z(H \cdot S)$ is a σ -martingale. If in addition $H \in \mathcal{A}$, then $Z(H \cdot S) \in \mathcal{M}_{\text{loc}}$.*

If $Z \in \mathcal{D}^{\text{weak}}$ and $H \in \mathcal{A}_1$, Lemma 4.6 shows that $Z(1 + H \cdot S)$ is a non-negative local martingale and, hence, a supermartingale. This means that Z is a *P-supermartingale density*, according to the terminology adopted in [3]. If we also have $Z_T > 0$ P -a.s., i.e., $Z \in \mathcal{D}$, then Z is an *equivalent supermartingale deflator* in the sense of Definition 4.9 of [38]. The importance of supermartingale densities/deflators has been first recognized by [41] in the context of utility maximisation.

We now show that the NSA condition ensures the existence of a tradable weak martingale deflator. This can already be guessed by carefully examining the proof of Theorem 4.3, but we prefer to give full details.

Proposition 4.7. *Let $\tau := \inf\{t \in [0, T] : \hat{K}_t = \infty\}$. If the NSA condition holds the process $\hat{Z} := \mathcal{E}(-\lambda \cdot M)\mathbf{1}_{[0, \tau)}$ is a tradable weak martingale deflator. Furthermore, $\hat{Z}N \in \mathcal{M}_{\text{loc}}$ for any $N = (N_t)_{0 \leq t \leq T} \in \mathcal{M}_{\text{loc}}$ orthogonal to M (in the sense of [32], Definition I.4.11).*

Proof. Note first that, due to Theorem 4.3, we have $\tau > 0$ P -a.s. Furthermore, the sequence $\{\tau_n\}_{n \in \mathbb{N}}$, defined as $\tau_n := \inf\{t \in [0, T] : \hat{K}_t \geq n\}$, $\forall n \in \mathbb{N}$, is an announcing sequence for τ , in the sense of I.2.16 of [32], and we have $[0, \tau) = \bigcup_{n \in \mathbb{N}} [0, \tau_n]$. Since $\hat{K}_{T \wedge \tau_n} \leq n$ P -a.s. for every $n \in \mathbb{N}$, the process $\hat{Z} := \mathcal{E}(-\lambda \cdot M)$ is well-defined as a continuous local martingale on $[0, \tau)$, in the sense of [46]. On $\{\tau \leq T\}$, we have $\hat{K}_\tau = \infty$ and $\hat{Z}_{\tau-} = 0$ P -a.s. Since $\hat{Z}^{\tau_n} \in \mathcal{M}^c$, for every $n \in \mathbb{N}$, Proposition 3.1 of [46] shows that \hat{Z} can be extended as a continuous local martingale on the whole interval $[0, T]$ by letting $\hat{Z} = \hat{Z}_{\tau-} = 0$ on $[\tau, T]$. Furthermore, the integration by parts formula gives that, for every $i = 1, \dots, d$:

$$\begin{aligned} d(\hat{Z}S^i)_t &= \hat{Z}_t dS^i_t + S^i_t d\hat{Z}_t + d\langle \hat{Z}, S^i \rangle_t = \hat{Z}_t dM^i_t + \hat{Z}_t d\langle M^i, M \rangle_t \lambda_t + S^i_t d\hat{Z}_t - \hat{Z}_t \lambda_t^\top d\langle M, M^i \rangle_t \\ &= \hat{Z}_t dM^i_t + S^i_t d\hat{Z}_t. \end{aligned}$$

Since S^i and \hat{Z} are continuous, this implies that $\hat{Z}S^i \in \mathcal{M}_{\text{loc}}^c$, for every $i = 1, \dots, d$. We have thus shown that $\hat{Z} = \mathcal{E}(-\lambda \cdot M)\mathbf{1}_{[0, \tau)} \in \mathcal{D}^{\text{weak}}$. To prove the tradability of \hat{Z} , note that the process $1/\hat{Z}$ is well defined on $[0, \tau) = \bigcup_{n \in \mathbb{N}} [0, \tau_n]$. Itô's formula gives then the following, for every $n \in \mathbb{N}$:

$$d\frac{1}{\hat{Z}_t^{\tau_n}} = -\frac{1}{(\hat{Z}_t^{\tau_n})^2} d\hat{Z}_t^{\tau_n} + \frac{1}{(\hat{Z}_t^{\tau_n})^3} d\langle \hat{Z} \rangle_t^{\tau_n} = \frac{1}{\hat{Z}_t^{\tau_n}} \lambda_t dM_t^{\tau_n} + \frac{1}{\hat{Z}_t^{\tau_n}} \lambda_t^\top d\langle M, M \rangle_t^{\tau_n} \lambda_t = \theta_t^n dS_t \quad (4.2)$$

where $\theta^n := \mathbf{1}_{[0, \tau_n]} \lambda \hat{Z}^{-1} \in \mathcal{A}_1$, for all $n \in \mathbb{N}$. Finally, for any $N = (N_t)_{0 \leq t \leq T} \in \mathcal{M}_{\text{loc}}$ orthogonal to M :

$$\hat{Z}N = N_0 + \hat{Z} \cdot N + N_- \cdot \hat{Z} + \langle \hat{Z}, N \rangle = N_0 + \hat{Z} \cdot N + N_- \cdot \hat{Z} - \hat{Z} \lambda \cdot \langle M, N \rangle = N_0 + \hat{Z} \cdot N + N_- \cdot \hat{Z}$$

where we have used the continuity of \hat{Z} and the orthogonality of M and N . Since \hat{Z} and N_- are predictable and locally bounded, being adapted and left-continuous, and since $N, \hat{Z} \in \mathcal{M}_{\text{loc}}$, Theorem IV.29 of [55] implies that $\hat{Z}N \in \mathcal{M}_{\text{loc}}$. \square

Remark 4.8 (*On the minimal martingale measure*). The process \hat{Z} is the candidate density process of the *minimal martingale measure*, originally introduced in [23] and defined as a probability measure $\hat{\mathbb{Q}} \sim P$ on (Ω, \mathcal{F}) with $\hat{\mathbb{Q}} = P$ on \mathcal{F}_0 such that S is a local $\hat{\mathbb{Q}}$ -martingale and every local P -martingale orthogonal to the martingale part M in the canonical decomposition of S (with respect to P) remains

a local \widehat{Q} -martingale. However, even if NSA holds, the process \widehat{Z} can fail to be a well-defined density process for two reasons. First, if $P(\widehat{Z}_T > 0) < 1$, the measure \widehat{Q} defined by $\frac{d\widehat{Q}}{dP} := \widehat{Z}_T$ fails to be equivalent to P , being only absolutely continuous. Second, \widehat{Z} may fail to be a true martingale, being instead a *strict* local martingale in the sense of [20], i.e., a local martingale which is not a true martingale, so that $E[\widehat{Z}_T] < E[\widehat{Z}_0] = 1$. In the latter case, \widehat{Q} fails to be a probability measure, since $\widehat{Q}(\Omega) = E[\widehat{Z}_T] < 1$.

We conclude this section by presenting two examples of financial markets that satisfy NIP but allow for strong arbitrage opportunities. In view of Theorem 4.3, the examples below satisfy $\nu_t = 0$ $P \otimes B$ -a.e. but $P(\alpha < T) > 0$.

Example 4.9. Let $M = (M_t)_{0 \leq t \leq T} \in \mathcal{M}_{\text{loc}}^c$ with $M_0 = 0$ and let τ be a stopping time such that $P(\tau < T) > 0$. Define the discounted price process $S = (S_t)_{0 \leq t \leq T}$ of a single risky asset as follows:

$$S = M + \langle M \rangle_{\cdot \wedge \tau}^\beta + (\langle M \rangle_{\cdot \vee \tau} - \langle M \rangle_\tau)^\gamma$$

for some $\gamma \leq 1/2 < \beta$. Then, due to Itô's formula:

$$dS_t = dM_t + \left(\beta \mathbf{1}_{\{t \leq \tau\}} \langle M \rangle_t^{\beta-1} + \gamma \mathbf{1}_{\{t > \tau\}} (\langle M \rangle_t - \langle M \rangle_\tau)^{\gamma-1} \right) d\langle M \rangle_t.$$

Theorem 3.1 implies that NIP holds. However, on $\{\tau < T\}$ we have that, for every $\varepsilon > 0$:

$$\widehat{K}_\tau^{\tau+\varepsilon} = \gamma^2 \int_\tau^{\tau+\varepsilon} (\langle M \rangle_t - \langle M \rangle_\tau)^{2(\gamma-1)} d\langle M \rangle_t = \begin{cases} \gamma^2 \log(\langle M \rangle_{\tau+t} - \langle M \rangle_\tau) \Big|_{t=\tau}^\varepsilon & \text{if } \gamma = 1/2 \\ \frac{\gamma^2}{2\gamma-1} (\langle M \rangle_{\tau+t} - \langle M \rangle_\tau)^{2\gamma-1} \Big|_{t=\tau}^\varepsilon & \text{if } \gamma < 1/2 \end{cases} = \infty.$$

This shows that in the present example we have $\alpha = \tau$ P -a.s. Hence, due to Theorem 4.3, the NSA condition fails to hold. By letting M be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, P)$, $\gamma = 1/2$ and $\tau = 0$, we recover the situation considered in Example 3.4 of [13].

Example 4.10. Let $W = (W_t)_{0 \leq t \leq T}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and define S as follows, for all $t \in [0, T]$:

$$S_t = W_t + \int_0^t \frac{W_u}{u} du.$$

Clearly, Theorem 3.1 implies that NIP holds. However, due to Corollary 3.2 of [34], we have $\int_0^\varepsilon (W_u/u)^2 du = \infty$ P -a.s. for every $\varepsilon > 0$, meaning that $\alpha = 0$ P -a.s. Theorem 4.3 then shows that NSA fails to hold.

5 No Arbitrage of the First Kind

An arbitrage of the first kind amounts to a non-negative and non-zero payoff which can be super-replicated via a non-negative portfolio by every market participant, regardless of his/her initial endowment. It is evident that a strong arbitrage opportunity yields an arbitrage of the first kind. Indeed, let $H \in \mathcal{A}_0$ generate a strong arbitrage opportunity and define $\xi := G_T(H)$. By Definition 2.3-(ii), it holds that $P(\xi \geq 0) = 1$ and $P(\xi > 0) > 0$. Moreover, for any $v \in (0, \infty)$, we also have $V_T(v, H) = v + G_T(H) > \xi$, thus showing that ξ yields an arbitrage of the first kind. A model satisfying NSA but allowing for arbitrages of the first kind will be presented at the end of this section, thus showing that NA1 is strictly stronger than NSA.

We now introduce two alternative notions of arbitrage which will be shown to be equivalent to an arbitrage of the first kind (Lemma 5.2).

Definition 5.1.

- (i) A sequence $\{H^n\}_{n \in \mathbb{N}} \subset \mathcal{A}_1$ generates an unbounded profit with bounded risk if the collection $\{G_T(H^n)\}_{n \in \mathbb{N}}$ is unbounded in probability, i.e., if $\lim_{m \rightarrow \infty} \sup_{n \in \mathbb{N}} P(G_T(H^n) > m) > 0$. If there exists no such sequence we say that the No Unbounded Profit with Bounded Risk (NUPBR) condition holds;
- (ii) let $\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ be a sequence such that $x_n \searrow 0$ as $n \rightarrow \infty$. A sequence $\{H^n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ with $H^n \in \mathcal{A}_{x_n}$, for all $n \in \mathbb{N}$, generates a cheap thrill if $V_T(x_n, H^n) \rightarrow \infty$ P -a.s. as $n \rightarrow \infty$ on some set with strictly positive probability. If there exists no such sequence we say that the No Cheap Thrill (NCT) condition holds.

The NUPBR condition has been first introduced under that name in [38] and corresponds to condition BK in [35]. The same condition also plays a key role in the seminal paper [11]. Note that there is no loss of generality in considering 1-admissible strategies in Definition 5.1-(i). Indeed, we have $\{G_T(H) : H \in \mathcal{A}_a\} = a\{G_T(H) : H \in \mathcal{A}_1\}$, for any $a > 0$, and, hence, the set of all final wealths generated by a -admissible strategies is bounded in probability if and only if the set of all final wealths generated by 1-admissible strategies is bounded in probability. The notion of cheap thrill has been introduced in [43] in the context of a complete Itô process model⁴.

The next lemma proves the equivalence between the notions introduced in Definition 5.1 and the notion of arbitrage of the first kind. The proof relies on techniques similar to those used in Section 3 of [11] and in the proof of Proposition 1 of [39]. Note that the proof of Lemma 5.2 does not rely on the continuity of the process S .

Lemma 5.2. *The NA1, NUPBR and NCT conditions are all equivalent.*

Proof. Let the random variable ξ generate an arbitrage of the first kind. By definition, for every $n \in \mathbb{N}$, there exists a strategy $H^n \in \mathcal{A}_{1/n}$ such that $V_T(1/n, H^n) \geq \xi$ P -a.s. For every $n \in \mathbb{N}$, define $\tilde{H}^n := nH^n$, so that $\{\tilde{H}^n\}_{n \in \mathbb{N}} \subset \mathcal{A}_1$ and $G_T(\tilde{H}^n) = nG_T(H^n) \geq n\xi - 1$ P -a.s. Since $P(\xi > 0) > 0$, this implies that the collection $\{G_T(\tilde{H}^n) : n \in \mathbb{N}\}$ is unbounded in probability.

Let $\{H^n\}_{n \in \mathbb{N}} \subset \mathcal{A}_1$ generate an unbounded profit with bounded risk, so that $P(G_T(H^n) \geq n) > \beta$ for all $n \in \mathbb{N}$ and for some $\beta > 0$. Let $\tilde{H}^n := n^{-1}H^n$, for every $n \in \mathbb{N}$, so that $\tilde{H}^n \in \mathcal{A}_{1/n}$ and $P(G_T(\tilde{H}^n) \geq 1) > \beta$. Let $f_n := n^{-1} + G_T(\tilde{H}^n) \geq 0$ P -a.s., for all $n \in \mathbb{N}$. Due to Lemma A1.1 of [11], there exists a sequence $\{g_n\}_{n \in \mathbb{N}}$, with $g_n \in \text{conv}\{f_n, f_{n+1}, \dots\}$, such that $\{g_n\}_{n \in \mathbb{N}}$ converges P -a.s. to a non-negative random variable g as $n \rightarrow \infty$. For all $n \in \mathbb{N}$, let K^n be the convex combination of strategies $\{\tilde{H}^m\}_{m \geq n}$ corresponding to g_n . It is easy to check that $K^n \in \mathcal{A}_{1/n}$, for every $n \in \mathbb{N}$. Furthermore, we have $G_T(K^n) = g_n + O(n^{-1})$, so that $G_T(K^n) \rightarrow g$ P -a.s. as $n \rightarrow \infty$. The last assertion of Lemma A1.1 of [11] implies that $P(g > 0) > 0$. By letting $x_n := \log(n)/n$ and $\tilde{K}^n := \log(n)K^n$, for every $n \in \mathbb{N}$, so that $\tilde{K}^n \in \mathcal{A}_{x_n}$, we then obtain a sequence $\{\tilde{K}^n\}_{n \in \mathbb{N}}$ which generates a cheap thrill.

Finally, let the sequence $\{H^n\}_{n \in \mathbb{N}}$ generate a cheap thrill, with respect to $\{x_n\}_{n \in \mathbb{N}}$. By definition, this implies that the set $C := \{V_T(x_n, H^n) : n \in \mathbb{N}\}$ is hereditarily unbounded in probability on

⁴We want to point out that a cheap thrill is equivalent to an *approximate arbitrage* in the sense of [10], as the reader can easily verify. However, we shall use the term “approximate arbitrage” with a different meaning in Section 6.

$\Omega_u := \{\omega \in \Omega : \lim_{n \rightarrow \infty} V_T(x_n, H^n)(\omega) = \infty\}$, in the sense of [4]. Then $\tilde{C} := \text{conv } C$ is hereditarily unbounded in probability as well and, by Lemma 2.3 of [4], for every $n \in \mathbb{N}$ there exists an element $f_n \in \tilde{C}$ such that $P(\Omega_u \cap \{f_n < 1\}) < P(\Omega_u)/2^{n+1}$. Let $A := \bigcap_{n \in \mathbb{N}} \{f_n \geq 1\}$ and $\xi := \mathbf{1}_A$. Then:

$$P(\Omega_u \setminus A) = P\left(\bigcup_{n \in \mathbb{N}} (\Omega_u \cap \{f_n < 1\})\right) \leq \sum_{n \in \mathbb{N}} P(\Omega_u \cap \{f_n < 1\}) < \sum_{n \in \mathbb{N}} \frac{P(\Omega_u)}{2^{n+1}} = \frac{P(\Omega_u)}{2}$$

which implies $P(A) > 0$, thus showing that $P(\xi \geq 0) = 1$ and $P(\xi > 0) > 0$. Note also that $\xi \leq \mathbf{1}_A f_n \leq f_n$ P -a.s., for every $n \in \mathbb{N}$. Since $f_n \in \text{conv}\{V_T(x_n, H^n) : n \in \mathbb{N}\}$, for every $n \in \mathbb{N}$, and $x_n \searrow 0$ as $n \rightarrow \infty$, this implies that ξ generates an arbitrage of the first kind. \square

Remark 5.3. We want to mention that the recent paper [28] provides a characterisation of the NA1 condition in terms of the equivalent *No Gratis Events (NGE)* condition. In particular, the NGE condition (and, consequently, the NA1 condition as well) is shown to be numéraire-independent. In the present setting, we shall give a simple proof of the latter property in Corollary 5.6.

The following theorem gives several equivalent characterisations of the NA1 condition (another equivalent and useful characterisation will be proved in the next section, see Corollary 6.7).

Theorem 5.4. *The following are equivalent, using the notation introduced in (2.1)-(2.2) and (4.1):*

- (i) any (and, consequently, all) of the NA1, NUPBR and NCT conditions holds;
- (ii) $\nu_t = 0$ $P \otimes B$ -a.e. and $\hat{K}_T = \int_0^T a_t^\top c_t^+ a_t dB_t < \infty$ P -a.s., i.e., $\lambda \in L_{\text{loc}}^2(M)$;
- (iii) there exists a tradable martingale deflator;
- (iv) $\mathcal{D} \neq \emptyset$, i.e., there exists a martingale deflator.

Proof. (i) \Rightarrow (ii): due to Lemma 5.2, the NA1, NUPBR and NCT conditions are equivalent. So, let us assume that NUPBR holds. Since NUPBR implies NSA, Theorem 4.3 gives that $\nu_t = 0$ $P \otimes B$ -a.e. and $\alpha = \inf\{t \in [0, T] : \hat{K}_t^{t+h} = \infty, \forall h \in (0, T-t]\} = \infty$ P -a.s. It remains to show that $\hat{K}_T < \infty$ P -a.s. Suppose on the contrary that $P(\tau \leq T) > 0$, where $\tau := \inf\{t \in [0, T] : \hat{K}_t = \infty\}$, so that $P(\hat{Z}_T = 0) = P(\hat{K}_T = \infty) > 0$, where the process \hat{Z} is defined as in Proposition 4.7. Define the sequence $\{\tau_n\}_{n \in \mathbb{N}}$ of stopping times $\tau_n := \inf\{t \in [0, T] : \hat{K}_t \geq n\}$, for every $n \in \mathbb{N}$. Clearly, we have $\tau_n \nearrow \tau$ P -a.s. as $n \rightarrow \infty$. As shown in equation (4.2), we have $\theta^n = \mathbf{1}_{(0, \tau_n]} \lambda \hat{Z}^{-1} \in \mathcal{A}_1$ and $G_T(\theta^n) = \hat{Z}_{T \wedge \tau_n}^{-1} - 1$, for every $n \in \mathbb{N}$, so that $G_T(\theta^n) \rightarrow \hat{Z}_{T \wedge \tau}^{-1} - 1$ P -a.s. as $n \rightarrow \infty$. Since $\hat{Z}_{T \wedge \tau} = 0$ on $\{\tau \leq T\}$ and $P(\tau \leq T) > 0$, this shows that $\{G_T(H^n) : n \in \mathbb{N}\}$ cannot be bounded in probability, thus contradicting the assumption that NUPBR holds.

(ii) \Rightarrow (iii): this follows directly from Proposition 4.7, since $\hat{K}_T < \infty$ P -a.s. implies $\tau = \infty$ P -a.s.

(iii) \Rightarrow (iv): due to Definition 4.4, this implication is trivial.

(iv) \Rightarrow (i): let $Z \in \mathcal{D}$ and suppose that the random variable ξ generates an arbitrage of the first kind, so that for every $v \in (0, \infty)$ there exists an element $H^v \in \mathcal{A}_v$ such that $V_T(v, H^v) \geq \xi$ P -a.s. Due to Lemma 4.6, the product $ZV(v, H^v) = Z(v + H^v \cdot S)$ is a non-negative local martingale and, hence, also a supermartingale. As a consequence, for every $v \in (0, \infty)$:

$$E[Z_T \xi] \leq E[Z_T V_T(v, H^v)] \leq E[Z_0 V_0(v, H^v)] = v.$$

Since $Z_T > 0$ P -a.s., this contradicts the assumption that $P(\xi > 0) > 0$. Due to Lemma 5.2, the NUPBR and NCT conditions hold as well. \square

Results related to Theorem 5.4 have already been obtained in Section 4 of [39] and in Section 3 of [30]. However, our proof is rather short and emphasises the tradability of the martingale deflator \widehat{Z} given in Proposition 4.7. In particular, we have shown that the event $\{\widehat{K}_T = \infty\} = \{\widehat{Z}_T = 0\}$ corresponds to the explosion of the final wealth generated by a sequence of 1-admissible strategies (see also Section 6 of [10] for a related discussion).

Remark 5.5 (*On the numéraire portfolio*). The NA1 condition can also be shown to be equivalent to the existence of the *numéraire portfolio*, defined as the strictly positive portfolio process $V^* = V(1, \theta^*)$, $\theta^* \in \mathcal{A}_1$, such that the ratio $V(1, \theta)/V^*$ is a supermartingale for all $\theta \in \mathcal{A}_1$ (see e.g. [3]). In the setting of the present paper, it is easy to verify that, as soon as NA1 holds, the numéraire portfolio coincides with the inverse of the tradeable martingale deflator \widehat{Z} , as a consequence of Theorem 5.4 together with Lemma 4.6 and Fatou's lemma (compare also with [30], Lemma 5). The equivalence between NUPBR and the existence of the numéraire portfolio is proved in full generality in [38] (see also [10] for related results).

As argued in Sections 3-4, the NIP and NSA conditions only exclude rather blatant forms of arbitrage, which can be thought of as pathologies of a financial market. The NIP and NSA conditions can therefore be regarded as indispensable “sanity checks”. As shown in Proposition 4.19 of [38], the failure of NUPBR (or, equivalently, of NA1 and NCT) precludes the solvability of any utility maximisation problem (in the case of a complete Itô-process model, an analogous result was already established in [43]). Moreover, in the recent paper [8] it has been proved that NUPBR is actually equivalent to a weak form of market viability, in the sense that any utility maximisation problem admits (locally) a solution if and only if NUPBR holds. As long as NA1 holds, the Benchmark Approach proposed by Eckhard Platen and collaborators provides a coherent framework for valuing contingent claims, see e.g. [51]-[54]. Always under the NA1 condition, valuation and hedging problems can be dealt with as in [22], [24] and [59]. Summing up, these facts suggest that NA1 can be regarded as the minimal no-arbitrage requirement for the purpose of financial modeling⁵.

An important property of the NA1 condition (as well as of NUPBR and NCT), which is not necessarily shared by stronger no-arbitrage conditions, is represented by its invariance with respect to a *change of numéraire*, as shown in the next corollary.

Corollary 5.6. *Let $V := V(1, \theta)$ be a P -a.s. strictly positive portfolio process, for some $\theta \in \mathcal{A}_1$. The NA1 condition holds (for S) if and only if the NA1 condition holds for $(S/V, 1/V)$.*

Proof. Due to Theorem 5.4, it suffices to show that $\mathcal{D} \neq \emptyset$ if and only if there exists a martingale deflator for $(S/V, 1/V)$. If $Z \in \mathcal{D} \neq \emptyset$, Lemma 4.6 implies that $Z' := ZV$ is a strictly positive local martingale with $Z'_0 = 1$. Since $Z'(S/V, 1/V) = Z(S, 1) \in \mathcal{M}_{\text{loc}}$, this shows that Z' is a martingale deflator for $(S/V, 1/V)$. Conversely, if Z' is a martingale deflator for $(S/V, 1/V)$ then $Z := Z'/V$ is a strictly positive local martingale with $Z_0 = 1$ and $ZS = Z'S/V \in \mathcal{M}_{\text{loc}}$, meaning that $Z \in \mathcal{D}$. \square

The next lemma describes the general structure of all martingale deflators. The result goes back to [62] and [9], but we give a short proof in the Appendix for the sake of completeness.

⁵In [39] it is shown that $\mathcal{D} \neq \emptyset$ is equivalent to the existence of a finitely additive measure Q on (Ω, \mathcal{F}) , weakly equivalent to P and locally countably additive, under which S has a sort of local martingale behavior (see also Section 5 of [5] for related results). We also want to mention that the equivalence (i) \Leftrightarrow (iv) in Theorem 5.4 has been recently established for general semimartingale models in the papers [40] and [66] (see also [10] for related results).

Lemma 5.7. *Suppose that any (and, consequently, all) of the NA1, NUPBR and NCT conditions holds. Then every martingale deflator $Z = (Z_t)_{0 \leq t \leq T}$ admits the following representation:*

$$Z = \mathcal{E}(-\lambda \cdot M + N) = \widehat{Z} \mathcal{E}(N)$$

for some $N = (N_t)_{0 \leq t \leq T} \in \mathcal{M}_{\text{loc}}$ with $N_0 = 0$, $\langle N, M \rangle = 0$ and $\Delta N > -1$ P -a.s. and where the process \widehat{Z} is defined as in Proposition 4.7.

Theorem 5.4 and Lemma 5.7 show that $\widehat{Z} = \mathcal{E}(-\lambda \cdot M)$ can be rightfully considered as the *minimal* (tradable) martingale deflator (compare also with Remark 4.8). Indeed, if \widehat{Z} fails to be a martingale deflator, i.e., if $P(\widehat{Z}_T = 0) > 0$, then there cannot exist any other martingale deflator.

We close this section by exhibiting a simple model which satisfies NSA but for which NA1 fails to hold. A similar example can also be found in Section 3.1 of [43].

Example 5.8. Let $W = (W_t)_{0 \leq t \leq T}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and define the process $X = (X_t)_{0 \leq t \leq T}$ as the solution to the following SDE, for some fixed $K > 0$:

$$dX_t = \frac{K - X_t}{T - t} dt + dW_t, \quad X_0 = 0.$$

The process X is a *Brownian bridge* (see [57], Exercise IX.2.12) starting at the level 0 and ending at the level $K > 0$. Let us define the discounted price process $S = (S_t)_{0 \leq t \leq T}$ of a single risky asset as $S_t := \exp(X_t)$, for all $t \in [0, T]$. Then, due to Itô's formula:

$$dS_t = S_t \left(\frac{K - \log(S_t)}{T - t} + \frac{1}{2} \right) dt + S_t dW_t, \quad S_0 = 1.$$

It is easy to see that the condition of Theorem 4.3 is satisfied and, hence, there are no strong arbitrage opportunities, since the process $\widehat{K} = \int_0^\cdot \left(\frac{K - \log(S_u)}{T - u} + \frac{1}{2} \right)^2 du$ does not jump to infinity. However, we have $\widehat{K}_t < \infty$ P -a.s. for all $t \in [0, T)$ but $\widehat{K}_T = \infty$ P -a.s. (compare also with [6]). Theorem 5.4 then implies that NA1 fails to hold.

6 No Arbitrage and No Free Lunch with Vanishing Risk

The goal of this section consists in studying the traditional NA and NFLVR conditions, on which the classical no-arbitrage pricing theory is based (we refer the reader to [17] for a complete account), and their relations with the weak no-arbitrage conditions discussed so far. As can be seen from Definition 2.3, the NA and NFLVR conditions exclude arbitrage possibilities that may require access to a finite line of credit and, hence, cannot be realized in an unlimited way by every market participant. This contrasts with the weak no-arbitrage conditions analyzed in the previous sections. Let us begin this section by introducing another (last) notion of arbitrage.

Definition 6.1. *A sequence $\{H^n\}_{n \in \mathbb{N}} \subset \mathcal{A}_c$, for some $c > 0$, generates an Approximate Arbitrage if $P(G_T(H^n) \geq 0) \rightarrow 1$ as $n \rightarrow \infty$ and there exists a constant $\delta > 0$ such that $P(G_T(H^n) > \delta) > \delta$. If there exists no such sequence we say that the No Approximate Arbitrage (NAA) condition holds.*

The notion of approximate arbitrage has been first introduced in [42] in the context of a complete Itô-process model and turns out to be equivalent to the notion of free lunch with vanishing risk introduced in Definition 2.3-(v), as shown in the next lemma, whose proof combines several

techniques already employed in [42] and [11]. Recall that $\mathcal{C} = (\{G_T(H) : H \in \mathcal{A}\} - L_+^0) \cap L^\infty$, according to the notation introduced at the end of Section 2.

Lemma 6.2. *The NFLVR condition and the NAA condition are equivalent.*

Proof. Suppose that NFLVR fails to hold. Then, as in Proposition 3.6 of [11], there exists either an arbitrage opportunity or a cheap thrill. Clearly, if there exists an arbitrage opportunity then there also exists an approximate arbitrage. We now show that the existence of a cheap thrill yields an approximate arbitrage, thus proving that NAA implies NFLVR. Due to Lemma 5.2 together with Theorem 5.4, the existence of a cheap thrill is equivalent to $0 < P(\widehat{K}_T = \infty) =: \delta$. For a fixed $K > 1 + \delta$ and for every $n \in \mathbb{N}$, define the stopping times

$$\sigma_n := \inf \left\{ t \in [0, T] : \widehat{Z}_t = 1/n \right\} \wedge T \quad \text{and} \quad \varrho_n := \inf \left\{ t \in [\sigma_n, T] : \widehat{Z}_t = \widehat{Z}_{\sigma_n}/K \right\} \wedge T$$

where \widehat{Z} is as in Proposition 4.7. For every $n \in \mathbb{N}$, define $Y^n := \widehat{Z}^{\sigma_n}/\widehat{Z}^{\varrho_n}$ and $H^n := \mathbf{1}_{[\sigma_n, \varrho_n]}(\widehat{Z}/\widehat{Z}_{\sigma_n})^{-1}\lambda$. Itô's formula implies then the following, for all $t \in [0, T]$:

$$Y_t^n = 1 + \int_0^t \frac{\widehat{Z}_{\sigma_n}}{\widehat{Z}_u} \mathbf{1}_{\{\sigma_n \leq u \leq \varrho_n\}} \lambda_u dM_u + \int_0^t \frac{\widehat{Z}_{\sigma_n}}{\widehat{Z}_u} \mathbf{1}_{\{\sigma_n \leq u \leq \varrho_n\}} \lambda_u^\top d\langle M, M \rangle_u \lambda_u = 1 + G_t(H^n)$$

thus showing that $\{H^n\}_{n \in \mathbb{N}} \subset \mathcal{A}_1$. Furthermore:

$$\{\widehat{K}_T = \infty\} = \{\widehat{Z}_T = 0\} \subseteq \{\sigma_n < \varrho_n < T\} \subseteq \{G_T(H^n) = \widehat{Z}_{\sigma_n}/\widehat{Z}_{\varrho_n} - 1\} = \{G_T(H^n) = K - 1\}$$

and hence $P(G_T(H^n) \geq K - 1) \geq \delta$. Since we have $\{G_T(H^n) < 0\} \subseteq \{\sigma_n < T\} \cap \{\widehat{K}_T < \infty\}$ and $P(\sigma_n < T, \widehat{K}_T < \infty) \rightarrow 0$ as $n \rightarrow \infty$, we also get

$$P(G_T(H^n) \geq 0) = 1 - P(G_T(H^n) < 0) \geq 1 - P(\sigma_n < T, \widehat{K}_T < \infty) \rightarrow 1$$

as $n \rightarrow \infty$, thus showing that the sequence $\{H^n\}_{n \in \mathbb{N}} \subset \mathcal{A}_1$ yields an approximate arbitrage.

Conversely, suppose that the sequence $\{H^n\}_{n \in \mathbb{N}} \subset \mathcal{A}_c$ generates an approximate arbitrage. By definition, for every $\varepsilon > 0$, we have $P(G_T(H^n)^- > \varepsilon) \leq P(G_T(H^n) < 0) \rightarrow 0$ as $n \rightarrow \infty$. This means that $G_T(H^n)^- \xrightarrow{P} 0$ as $n \rightarrow \infty$ and, passing to a subsequence, we can assume that the convergence takes place P -a.s. For every $n \in \mathbb{N}$, let $f_n := G_T(H^n) \wedge \delta \in \mathcal{C}$, so that $P(f_n = \delta) > \delta$ and $f_n^- \rightarrow 0$ P -a.s. as $n \rightarrow \infty$. Due to Lemma A1.1 (and the subsequent Remark 2) of [11], there exists a sequence $\{g_n\}_{n \in \mathbb{N}}$, with $g_n \in \text{conv}\{f_n, f_{n+1}, \dots\}$, such that $g_n \rightarrow g$ P -a.s. as $n \rightarrow \infty$ for some random variable $g : \Omega \rightarrow [0, \delta]$ with $P(g > 0) > 0$. More precisely, due to the bounded convergence theorem:

$$\delta P(g > 0) \geq E[g \mathbf{1}_{\{g > 0\}}] = E[g] = \lim_{n \rightarrow \infty} E[g_n] \geq \delta^2$$

meaning that $\beta := P(g > 0) \geq \delta > 0$. Egorov's theorem gives that g_n converges to g as $n \rightarrow \infty$ uniformly on a set Ω' with $P(\Omega') \geq 1 - \beta/2$. For every $n \in \mathbb{N}$, define $h_n := g_n \wedge \delta \mathbf{1}_{\Omega'}$, so that $\{h_n\}_{n \in \mathbb{N}} \in \mathcal{C}$ and $h_n \rightarrow g \mathbf{1}_{\Omega'}$ in the norm topology of L^∞ , i.e., $g \mathbf{1}_{\Omega'} \in \overline{\mathcal{C}} \cap L_+^\infty$. Since

$$P(g \mathbf{1}_{\Omega'} > 0) = 1 - P(\{g = 0\} \cup \Omega'^c) \geq P(\Omega') - P(g = 0) \geq 1 - \beta/2 - (1 - \beta) = \beta/2 > 0,$$

this shows that NFLVR fails to hold, thus proving that NFLVR implies NAA. \square

Before formulating the next theorem, we need to recall the classical and well-known notion of *Equivalent Local Martingale Measure*.

Definition 6.3. A probability measure Q on (Ω, \mathcal{F}) with $Q \sim P$ is said to be an Equivalent Local Martingale Measure (ELMM) for S if S is a local Q -martingale.

Remark 6.4 (*On martingale deflators and ELMMs*). Suppose that there exists an ELMM Q for S . Letting $Z^Q = (Z_t^Q)_{0 \leq t \leq T}$ be its density process, Bayes' rule implies that $Z^Q S \in \mathcal{M}_{\text{loc}}$, meaning that $Z^Q/Z_0^Q \in \mathcal{D}$. Conversely, in view of Remark 4.5, an element $Z \in \mathcal{D}$ can be taken as the density process of an ELMM if and only if $E[Z_T] = 1$.

Theorem 6.5. The following are equivalent, using the notation introduced in (2.1)-(2.2) and (4.1):

- (i) the NFLVR condition holds;
- (ii) there exists an ELMM for S ;
- (iii) $\nu_t = 0$ $P \otimes B$ -a.e., $\widehat{K}_T = \int_0^T a_t^\top c_t^+ a_t dB_t < \infty$ P -a.s. and there exists $N = (N_t)_{0 \leq t \leq T} \in \mathcal{M}_{\text{loc}}$ with $N_0 = 0$, $\langle N, M \rangle = 0$ and $\Delta N > -1$ P -a.s. such that $\widehat{Z}\mathcal{E}(N) \in \mathcal{M}$;
- (iv) the conditions NA1 (or, equivalently, NUPBR/NCT) and NA both hold;
- (v) the NAA condition holds.

Proof. (i) \Leftrightarrow (ii): this is the content of Corollary 1.2 of [11], recalling that S is a continuous semimartingale and, hence, a locally bounded semimartingale.

(ii) \Leftrightarrow (iii): this equivalence follows from Theorem 5.4 and Lemma 5.7 together with Remark 6.4.

(ii) \Rightarrow (iv): the existence of an ELMM for S implies that $\mathcal{D} \neq \emptyset$. Hence, due to Theorem 5.4, the NA1 condition (as well as NUPBR and NCT) holds. Let $H \in \mathcal{A}$ yield an arbitrage opportunity. Lemma 4.6 and Bayes' rule imply that the process $G(H)$ is a local Q -martingale uniformly bounded from below. Due to Fatou's lemma, it is also a Q -supermartingale and, hence, $E^Q[G_T(H)] \leq 0$. Clearly, this contradicts the assumption that H yields an arbitrage opportunity.

(iv) \Rightarrow (v): this implication follows as in the first part of the proof of Lemma 6.2.

(v) \Leftrightarrow (i): this follows from Lemma 6.2. □

Remarks 6.6. 1) As can be seen from condition (iii) of Theorem 6.5, the NFLVR condition, unlike the weak no-arbitrage conditions discussed in the previous sections, does not only depend on the (properties of the) characteristics of the discounted price process S but also on the structure of the underlying filtration \mathbb{F} . In particular, this means that in general one cannot construct an arbitrage opportunity (or a free lunch with vanishing risk) by relying on the characteristics of the price process only (to this effect, compare also [38], Example 4.7)⁶.

2) There is no general relation between NA1 and NA. Indeed, as shown in the example at the end of this section, it might well be that NA1 holds but nevertheless there exist arbitrage opportunities. Conversely, it is possible to construct a model which admits no arbitrage opportunities but does not satisfy NA1 (an explicit example can be found in Section 4 of [42]; see also [29], Example 1.37).

⁶We want to mention that in some special cases it is possible to check the NFLVR condition in terms of the characteristics of the discounted price process S . For instance, in the case when S is strictly positive and one can take $dB_t = dt$ in (2.1), a probabilistic characterisation of the absence of arbitrage opportunities in terms of the characteristics of S has been obtained in the recent paper [45]. Also, in the case of non-negative one-dimensional Markovian diffusions, necessary and sufficient conditions for the validity of NFLVR are provided in [49].

3) We want to warn the reader that the validity of the NFLVR condition does not ensure that the measure \widehat{Q} defined by $d\widehat{Q}/dP := \widehat{Z}_T$ is an ELMM for S , since NFLVR does not imply in general that $E[\widehat{Z}_T] = 1$. In view of Remark 4.8, this amounts to saying that NFLVR does not guarantee the existence of the minimal martingale measure (an explicit counterexample is provided in [15]). In other words, the NFLVR condition cannot be checked by looking only at the properties of the minimal (weak) martingale deflator \widehat{Z} , unlike the NA1 condition.

The following corollary gives an interesting alternative characterisation of the NA1 condition, which complements Theorem 5.4.

Corollary 6.7. *The NA1 condition holds if and only if there exists a P -a.s. strictly positive portfolio process $V := V(1, \theta)$, for some $\theta \in \mathcal{A}_1$, such that the NFLVR condition holds for $(S/V, 1/V)$.*

Proof. Due to Theorem 5.4, the NA1 condition implies the existence of a tradable martingale deflator Z , so that $1/Z = V(1, \theta)$ for some $\theta \in \mathcal{A}_1$ (see also Remark 4.5). By letting $V := V(1, \theta)$, this means that $1/V \in \mathcal{M}_{\text{loc}}$ and $S/V \in \mathcal{M}_{\text{loc}}$ and so P is an ELMM for $(S/V, 1/V)$. Due to Theorem 6.5, this implies that $(S/V, 1/V)$ satisfies NFLVR. Conversely, if NFLVR holds for $(S/V, 1/V)$, Theorem 6.5 gives the existence of an ELMM Q for $(S/V, 1/V)$, with density process Z^Q . By Bayes' rule, we have $Z^Q/V \in \mathcal{M}_{\text{loc}}$ and $Z^Q S/V \in \mathcal{M}_{\text{loc}}$. This means that $Z := Z^Q/(Z_0^Q V) \in \mathcal{D}$. Theorem 5.4 then implies that NA1 holds for S . \square

In particular, the above corollary shows that, as soon as NA1 holds, we can find a suitable *numéraire* V such that the classical and stronger NFLVR condition holds in the V -discounted financial market $(S/V, 1/V)$, regardless of the validity of NFLVR for the original financial market. In particular, if NA1 holds, the process \widehat{Z} is a tradable martingale deflator and, hence, letting $\widehat{V} := 1/\widehat{Z}$, the NFLVR condition holds for $(S/\widehat{V}, 1/\widehat{V})$. This suggests that, even in the absence of an ELMM for S , the financial market $(S/\widehat{V}, 1/\widehat{V})$ can be regarded as a natural setting for solving pricing and portfolio optimisation problems, as proposed in the context of the Benchmark Approach (see [54], Chapter 10).

We close this section with an example of a class of models that satisfy NA1 but for which NFLVR fails to hold. In particular, this generalizes the typical example based on a three-dimensional Bessel process (see [14], Corollary 2.10). Other financial models which satisfy NA1 but not NFLVR can be found in the context of Stochastic Portfolio Theory (see e.g. [22], Sections 5-6) and within the Benchmark Approach (see e.g. [51], [54], Chapters 12-13, and [29], Chapter 5).

Example 6.8. Let $W = (W_t)_{0 \leq t \leq T}$ be a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}_T^W, \mathbb{F}^W, P)$, with $\mathbb{F}^W = (\mathcal{F}_t^W)_{0 \leq t \leq T}$ denoting the P -augmented natural filtration of W , and take a continuous function $\sigma : (0, \infty) \rightarrow (0, \infty)$ such that the following SDE admits a unique strong solution:

$$dS_t = S_t \sigma^2(S_t) dt + S_t \sigma(S_t) dW_t, \quad S_0 = 1. \quad (6.1)$$

Assume furthermore that $\int_x^\infty \frac{1}{y\sigma^2(1/y)} dy < \infty$ for some $x \in (0, \infty)$. According to the notation introduced in Section 2, we have $A_t = \int_0^t S_u \sigma^2(S_u) du$ and $M_t = \int_0^t S_u \sigma(S_u) dW_u$, for all $t \in [0, T]$, and $\lambda = 1/S$. Since S is locally bounded, this implies that $\widehat{K}_T = \int_0^T \lambda_t^2 d\langle M \rangle_t < \infty$ P -a.s., thus showing that NA1 holds (see Theorem 5.4). Since W enjoys the martingale representation property in the filtration \mathbb{F}^W , Lemma 5.7 implies that $\mathcal{D} = \{\widehat{Z}\}$, where $\widehat{Z} = \mathcal{E}(-\lambda \cdot M) = \mathcal{E}(-\int \sigma(S) dW) = 1/S$. However, since $\int_x^\infty \frac{1}{y\sigma^2(1/y)} dy < \infty$, for some $x \in (0, \infty)$, Corollary 4.3 of [48] shows that \widehat{Z} is a *strict*

local martingale in the sense of [20], i.e., it is a local martingale which is not a true martingale, so that $E[\widehat{Z}_T] < 1$. Due to Theorem 6.5, this shows that NFLVR fails for the model (6.1).

In the context of the present example, it is easy to construct explicitly an arbitrage opportunity. Indeed, let us define the process $L = (L_t)_{0 \leq t \leq T}$ by $L_t := E[\widehat{Z}_T | \mathcal{F}_t]$, for all $t \in [0, T]$. Then, due to the martingale representation property, there exists an \mathbb{F}^W -predictable process $\theta = (\theta_t)_{0 \leq t \leq T}$ with $\int_0^T \theta_t^2 dt < \infty$ P -a.s such that $L = v + \theta \cdot W$ P -a.s. Let us also define the process $V := L/\widehat{Z} = LS$. A simple application of the product rule gives

$$dV_t = L_t dS_t + S_t dL_t + d\langle L, S \rangle_t = L_t S_t (\sigma^2(S_t) dt + \sigma(S_t) dW_t) + S_t \theta_t dW_t + S_t \sigma(S_t) \theta_t dt = \varphi_t dS_t$$

where the process $\varphi = (\varphi_t)_{0 \leq t \leq T}$ is defined as $\varphi_t := L_t + \theta_t / \sigma(S_t)$, for all $t \in [0, T]$. The continuity of L , S and of the function $\sigma(\cdot)$ implies that $\varphi \in L(S)$. Noting that $G(\varphi) = V - V_0 \geq -E[\widehat{Z}_T] > -1$ P -a.s., we also have $\varphi \in \mathcal{A}_1$. Since $G_T(\varphi) = V_T - V_0 = 1 - E[\widehat{Z}_T] > 0$ P -a.s., this means that φ yields an arbitrage opportunity. We have thus shown that the model (6.1) allows for the possibility of replicating a risk-free zero-coupon bond of unitary nominal value starting from an initial investment which is strictly less than one. However, not every market participant can profit from this arbitrage opportunity in an unlimited way, since the strategy $\varphi \in \mathcal{A}_1$ requires a non-negligible line of credit.

In particular, any function of the form $\sigma(x) = x^\mu$, for $\mu < 0$, satisfies the integrability condition $\int_x^\infty \frac{1}{y\sigma^2(1/y)} dy < \infty$ for any $x \in (0, \infty)$. In the special case $\mu = -1$, it can be shown that the process S is a three-dimensional Bessel process (see [57], Chapter XI), the classical example of a financial model for which NFLVR fails, as shown already in [12], in Corollary 2.10 of [14] and in Example 4.6 of [38]. Related results can also be found in [18].

7 Conclusions

We close the paper with the following table, which summarises the no-arbitrage conditions introduced in Definition 2.3 and studied so far, together with their characterisations (see Theorems 3.1, 4.3, 5.4 and 6.5) and their equivalent formulations (see Lemmata 4.2, 5.2 and 6.2).

CONDITION	PROBABILISTIC CHARACTERISATION	EQUIVALENT FORMULATION
<i>No Increasing Profit (NIP)</i>	$\nu_t = 0$ $P \otimes B$ -a.e.	-
<i>No Strong Arbitrage (NSA)</i>	$\nu_t = 0$ $P \otimes B$ -a.e. and $\alpha = \infty$ P -a.s. (i.e., \widehat{K} does not jump to infinity)	<i>No Immediate Arbitrage (NIA)</i>
<i>No Arbitrage of the First Kind (NA1)</i>	$\nu_t = 0$ $P \otimes B$ -a.e. and $\widehat{K}_T < \infty$ P -a.s.	<i>No Unbounded Profit with Bounded Risk (NUPBR)</i> <i>No Cheap Thrill (NCT)</i>
<i>No Free Lunch with Vanishing Risk (NFLVR)</i>	$\nu_t = 0$ $P \otimes B$ -a.e. and $\widehat{K}_T < \infty$ P -a.s. and $\exists N \in \mathcal{M}_{\text{loc}}$ with $N_0 = 0$, $\langle N, M \rangle = 0$, $\Delta N > -1$ P -a.s. such that $\widehat{Z}\mathcal{E}(N) \in \mathcal{M}$.	<i>No Approximate Arbitrage (NAA)</i>

Acknowledgements: The author is thankful to Wolfgang J. Runggaldier for useful remarks and to Monique Jeanblanc for several discussions on the topic of the present paper. Financial support from the *Chaire Risque de Cr dit* (F d ration Bancaire Fran aise) is gratefully acknowledged.

A Appendix

Proof of Lemma 4.6.

The first part of the proof relies on arguments similar to those used in the proofs of Proposition 3.2 of [25] and Proposition 8 of [58]. Let $Z = (Z_t)_{0 \leq t \leq T} \in \mathcal{D}^{\text{weak}}$ and $H \in L(S)$. Define the \mathbb{R}^{d+1} -valued local martingale $Y = (Y_t)_{0 \leq t \leq T}$ by $Y_t := (Z_t S_t^1, \dots, Z_t S_t^d, Z_t)^\top$ and let $L(Y)$ be the set of all \mathbb{R}^{d+1} -valued predictable Y -integrable processes, in the sense of Definition III.6.17 of [32]. For all $n \in \mathbb{N}$, define also $H(n) := H \mathbf{1}_{\{\|H\| \leq n\}}$. Then, using twice the integration by parts formula and the associativity of the stochastic integral:

$$\begin{aligned} Z(H(n) \cdot S) &= Z_- \cdot (H(n) \cdot S) + (H(n) \cdot S)_- \cdot Z + [Z, H(n) \cdot S] \\ &= (Z_- H(n)) \cdot S + (H(n) \cdot S)_- \cdot Z + H(n) \cdot [S, Z] \\ &= H(n) \cdot (Z_- \cdot S) + (H(n) \cdot S)_- \cdot Z + H(n) \cdot [S, Z] \\ &= H(n) \cdot (ZS - S_- \cdot Z) + (H(n) \cdot S)_- \cdot Z \\ &= H(n) \cdot (ZS) + \left((H(n) \cdot S)_- - H(n)^\top S_- \right) \cdot Z = K(n) \cdot Y \end{aligned}$$

where, for every $n \in \mathbb{N}$, the \mathbb{R}^{d+1} -valued predictable process $K(n)$ is defined as $K(n)^i := H(n)^i$, for all $i = 1, \dots, d$, and $K(n)^{d+1} := (H(n) \cdot S)_- - H(n)^\top S_-$. Clearly, we have $K(n) \in L(Y)$, since $K(n)$ is predictable and locally bounded, for every $n \in \mathbb{N}$. Define the \mathbb{R}^{d+1} -valued predictable process K by $K^i := H^i$, for all $i = 1, \dots, d$, and $K^{d+1} := (H \cdot S)_- - H^\top S_-$. Since $H \in L(S)$, $H(n) \cdot S$ converges to $H \cdot S$ in the Emery topology of semimartingales as $n \rightarrow \infty$. This implies that $K(n) \cdot Y = Z(H(n) \cdot S)$ also converges in Emery's topology, since the multiplication with Z is a continuous operation. Since the space $\{K \cdot Y : K \in L(Y)\}$ is closed in Emery's topology (see [32], Proposition III.6.26), we can conclude that $Z(H \cdot S) = \bar{K} \cdot Y$ for some $\bar{K} \in L(Y)$. But since $K(n)$ converges to K (P -a.s. uniformly in t , at least along a subsequence) as $n \rightarrow \infty$, we can conclude that $\bar{K} = K$ (see [47]). This shows that $K \in L(Y)$. Since $Y \in \mathcal{M}_{\text{loc}}$ and $K \in L(Y)$, Proposition III.6.42 of [32] implies that $Z(H \cdot S) = K \cdot Y$ is a σ -martingale. To prove the second assertion of the lemma, suppose that $H \in \mathcal{A}$, i.e., there exists a positive constant a such that $(H \cdot S)_t \geq -a$ P -a.s. for all $t \in [0, T]$. The process $Z(a + H \cdot S)$ is a σ -martingale, being the sum of a local martingale and a σ -martingale. Furthermore, Proposition 3.1 and Corollary 3.1 of [37] imply that $Z(a + H \cdot S)$ is a supermartingale, being a non-negative σ -martingale, and, hence, also a local martingale (compare also with [2], Corollary 3.5). In turn, this implies that $Z(H \cdot S) \in \mathcal{M}_{\text{loc}}$, being the difference of two local martingales.

Proof of Lemma 5.7.

Let $Z = (Z_t)_{0 \leq t \leq T} \in \mathcal{D}$. By Definition 4.4 and Remark 4.5, the process $Z \in \mathcal{M}_{\text{loc}}$ satisfies $P(Z_t > 0 \text{ and } Z_{t-} > 0 \text{ for all } t \in [0, T]) = 1$. In view of Theorem II.8.3 of [32], the stochastic logarithm $L := Z^{-1} \cdot Z$ is well-defined as a local martingale with $L_0 = 0$ and satisfies $Z = \mathcal{E}(L)$. Since $M \in \mathcal{M}_{\text{loc}}^c$, the process L admits a Galtchouk-Kunita-Watanabe decomposition with respect

to M , see [1]. So, we can write $L = \psi \cdot M + N$ for some \mathbb{R}^d -valued predictable process $\psi = (\psi_t)_{0 \leq t \leq T} \in L_{\text{loc}}^2(M)$ and for some $N = (N_t)_{0 \leq t \leq T} \in \mathcal{M}_{\text{loc}}$ with $N_0 = 0$ and $\langle N, M \rangle = 0$. Then, for all $i = 1, \dots, d$:

$$\begin{aligned} ZS^i &= Z_- \cdot S^i + S^i \cdot Z + \langle Z, S^i \rangle = Z_- \cdot A^i + Z_- \cdot M^i + S^i \cdot Z + \langle Z, M^i \rangle \\ &= Z_- \cdot \left(\int d\langle M^i, M \rangle \lambda \right) + Z_- \cdot M^i + S^i \cdot Z + Z_- \cdot \langle \psi \cdot M + N, M^i \rangle \\ &= Z_- \cdot \left(\int d\langle M^i, M \rangle (\lambda + \psi) \right) + Z_- \cdot M^i + S_-^i \cdot Z \end{aligned}$$

By Theorem IV.29 of [55], we have $Z_- \cdot M^i \in \mathcal{M}_{\text{loc}}^c$ and $S_-^i \cdot Z \in \mathcal{M}_{\text{loc}}$. In turn, this implies that $Z_- \cdot \left(\int d\langle M^i, M \rangle (\lambda + \psi) \right) \in \mathcal{M}_{\text{loc}}^c$, for all $i = 1, \dots, d$. Since $Z_- > 0$ P -a.s., Theorem III.15 of [55] allows to conclude that $\int d\langle M^i, M \rangle (\lambda + \psi) = 0$ for all $i = 1, \dots, d$, which in turn implies that the stochastic integral $\psi \cdot M$ is indistinguishable from $-\lambda \cdot M$, thus yielding the following representation:

$$Z = \mathcal{E}(L) = \mathcal{E}(\psi \cdot M + N) = \mathcal{E}(-\lambda \cdot M + N) = \widehat{Z} \mathcal{E}(N)$$

where the last equality follows by Yor's formula (see e.g. [55], Theorem II.38) and Proposition 4.7. Since $Z > 0$ and $\widehat{Z} > 0$ P -a.s., we also have $\mathcal{E}(N) > 0$ P -a.s., meaning that $\Delta N > -1$ P -a.s.

References

- [1] ANSEL, J.P. AND STRICKER, C. (1993). Décomposition de Kunita-Watanabe. *Séminaire de Probabilités XXVII*, 30–32, Lecture Notes in Mathematics vol. 1557, Springer.
- [2] ANSEL, J.P. AND STRICKER, C. (1994). Couverture des actifs contingents et prix maximum. *Annales de l'Institut Henri Poincaré* **30**, 303–315.
- [3] BECHERER, D. (2001). The numeraire portfolio for unbounded semimartingales. *Financ. Stoch.* **5**, 327–341.
- [4] BRANNATH, W. AND SCHACHERMAYER, W. (1999). A bipolar theorem for $L_+^0(\Omega, \mathcal{F}, P)$. *Séminaire de Probabilités XXXIII*, 349–354, Lecture Notes in Mathematics vol. 1709, Springer.
- [5] CASSESE, G. (2005). A note on asset bubbles in continuous-time. *Int. J. Theor. Appl. Financ.* **8**, 523–536.
- [6] CHENG, S.T. (1991). On the feasibility of arbitrage-based option pricing when stochastic bond price processes are involved. *J. Econ. Theory* **53**, 185–198.
- [7] CHOU, C.S., MEYER, P.A. AND STRICKER, C. (1980). Sur les intégrales stochastiques de processus prévisibles non bornés. *Séminaire de Probabilités XIV*, 128–139, Lecture Notes in Mathematics vol. 784, Springer.
- [8] CHOULLI, T., DENG, J. AND MA, J. (2012). The fundamental theorem of utility maximization and numéraire portfolio. Preprint (available on <http://arxiv.org/abs/1211.4598>).
- [9] CHOULLI, T. AND STRICKER, C. (1996). Deux applications de la décomposition de Galtchouk-Kunita-Watanabe. *Séminaire de Probabilités XXX*, 12–23, Lecture Notes in Mathematics vol. 1626, Springer.
- [10] CHRISTENSEN, M.M. AND LARSEN, K. (2007). No arbitrage and the growth optimal portfolio. *Stoch. Anal. Appl.* **25**, 255–280.

- [11] DELBAEN, F. AND SCHACHERMAYER, W. (1994). A general version of the fundamental theorem of asset pricing. *Math. Ann.* **300**, 463–520.
- [12] DELBAEN, F. AND SCHACHERMAYER, W. (1995a). Arbitrage possibilities in Bessel processes and their relations to local martingales. *Prob. Theory Rel. Fields* **102**, 357–366.
- [13] DELBAEN, F. AND SCHACHERMAYER, W. (1995b). The existence of absolutely continuous local martingale measures. *Ann. Appl. Prob.* **5**, 926–945.
- [14] DELBAEN, F. AND SCHACHERMAYER, W. (1995c). The no-arbitrage property under a change of numéraire. *Stochastics and Stoch. Rep.* **53**, 213–226.
- [15] DELBAEN, F. AND SCHACHERMAYER, W. (1998a). A simple counterexample to several problems in the theory of asset pricing. *Math. Financ.* **8**, 1–12.
- [16] DELBAEN, F. AND SCHACHERMAYER, W. (1998b). The fundamental theorem of asset pricing for unbounded stochastic processes. *Math. Ann.* **312**, 215–250.
- [17] DELBAEN, F. AND SCHACHERMAYER, W. (2006). *The Mathematics of Arbitrage*. Springer, Berlin - Heidelberg - New York.
- [18] DELBAEN, F. AND SHIRAKAWA, H. (2002). No arbitrage condition for positive diffusion price processes. *Asia Pac. Financ. Markets* **9**: 159–168.
- [19] DZHAPARIDZE, K. AND SPREIJ, P. (1993). On correlation calculus for multivariate martingales. *Stoch. Proc. Appl.* **46**, 283–299.
- [20] ELWORTHY, K.D., LI, X.M. AND YOR, M. (1999). The importance of strictly local martingales; applications to radial Ornstein-Uhlenbeck processes. *Prob. Theory Rel. Fields* **115**, 325–355.
- [21] FERNHOLZ, R. (2002). *Stochastic Portfolio Theory*. Springer, New York.
- [22] FERNHOLZ, R. AND KARATZAS, I. (2009). Stochastic portfolio theory: an overview. In *Mathematical Modeling and Numerical Methods in Finance*, eds. A. Bensoussan and Q. Zhang. Handbook of Numerical Analysis **XV**, 89–167, North-Holland, Oxford.
- [23] FÖLLMER, H. AND SCHWEIZER, M. (1991). Hedging of contingent claims under incomplete information. In *Applied Stochastic Analysis*, eds. M.H.A. Davis and R.J. Elliott. Stochastic monographs **5**, 389–414, Gordon and Breach, London - New York.
- [24] FONTANA, C. AND RUNGALDIER, W.J. (2013). Diffusion-based models for financial markets without martingale measures. In *Risk Measures and Attitudes*, eds. F. Biagini, A. Richter and H. Schlesinger, 45–81. EAA Lecture Notes, Springer, London.
- [25] GOURIEROUX, C., LAURENT, J.P. AND PHAM, H. (1998). Mean-variance hedging and numéraire. *Math. Financ.* **8**, 179–200.
- [26] HARRISON, J.M. AND KREPS, D.M. (1979). Martingales and arbitrage in multiperiod securities markets. *J. Econ. Theory* **20**, 381–408.
- [27] HARRISON, J.M. AND PLISKA, S.R. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stoch. Proc. Appl.* **11**, 215–260.
- [28] HERDEGEN, M. (2012). No-arbitrage in a numéraire independent modelling framework. NCCR FINRISK working paper no. 775.
- [29] HULLEY, H. (2009). *Strict Local Martingales in Continuous Financial Market Models*. PhD thesis, University of Technology Sydney.

- [30] HULLEY, H. AND SCHWEIZER, M. (2010). M^6 - on minimal market models and minimal martingale measures. In *Contemporary Quantitative Finance: Essays in Honour of Eckhard Platen*, eds. C. Chiarella and A. Novikov, 35–51. Springer, Berlin - Heidelberg.
- [31] JACOD, J. (1980). Intégrales stochastiques par rapport à une semi-martingale vectorielle et changements de filtration. *Séminaire de Probabilités XIV*, 161–172, Lecture Notes in Mathematics vol. 784, Springer.
- [32] JACOD, J. AND SHIRYAEV, A.N. (2003). *Limit Theorems for Stochastic Processes*, 2nd edn. Springer, Berlin - Heidelberg - New York.
- [33] JARROW, R. AND PROTTER, P. (2005). Large traders, hidden arbitrage, and complete markets. *J. Bank. Financ.* **29**, 2803–2820.
- [34] JEULIN, T. AND YOR, M. (1979). Inégalité de Hardy, semimartingales, et faux-amis. *Séminaire de Probabilités XIII*, 332–359, Lecture Notes in Mathematics vol. 721, Springer.
- [35] KABANOV, Y. (1997). On the FTAP of Kreps-Delbaen-Schachermayer. In *Statistics and Control of Stochastic Processes: The Liptser Festschrift*, eds. Y. Kabanov, B.L. Rozovskii, A.N. Shiryaev, 191–203. World Scientific, Singapore.
- [36] KABANOV, Y. AND STRICKER, C. (2005). Remarks on the true no-arbitrage property. *Séminaire de Probabilités XXXVIII*, 186–194, Lecture Notes in Mathematics vol. 1857, Springer.
- [37] KALLSEN, J. (2004). σ -localization and σ -martingales. *Theory Prob. Appl.* **48**, 152–163.
- [38] KARATZAS, I. AND KARDARAS, K. (2007). The numeraire portfolio in semimartingale financial models. *Financ. Stoch.* **11**, 447–493.
- [39] KARDARAS, C. (2010). Finitely additive probabilities and the fundamental theorem of asset pricing. In *Contemporary Quantitative Finance: Essays in Honour of Eckhard Platen*, eds. C. Chiarella and A. Novikov, 19–34. Springer, Berlin - Heidelberg.
- [40] KARDARAS, C. (2012). Market viability via absence of arbitrage of the first kind. *Financ. Stoch.* **16**, 651–667.
- [41] KRAMKOV, D. AND SCHACHERMAYER, W. (1999). The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Ann. Appl. Prob.* **9**, 904–950.
- [42] LEVENTAL, S. AND SKOROHOD, A.S. (1995). A necessary and sufficient condition for absence of arbitrage with tame portfolios. *Ann. Appl. Prob.* **5**, 906–925.
- [43] LOEWENSTEIN, M. AND WILLARD, G.A. (2000). Local martingales, arbitrage, and viability. *Econ. Theory* **16**, 135–161.
- [44] LONDONO, J.A. (2004). State tameness: a new approach for credit constraints. *Electron. Commun. Prob.* **9**, 1–13.
- [45] LYASOFF, A. (2012). The two fundamental theorems of asset pricing for a class of continuous-time financial markets. To appear in: *Math. Financ.*
- [46] MAISONNEUVE, B. (1977). Une mise au point sur les martingales locales continues définies sur un intervalle stochastique. *Séminaire de Probabilités XI*, 435–445, Lecture Notes in Mathematics vol. 581, Springer.
- [47] MÉMIN, J. (1980). Espaces des semimartingales et changement de probabilité. *Z. Wahrscheinlichkeit.* **52**, 9–39.

- [48] MIJATOVIĆ, A. AND URUSOV, M. (2012a). On the martingale property of certain local martingales. *Prob. Theory Rel. Fields.* **152**, 1–30.
- [49] MIJATOVIĆ, A. AND URUSOV, M. (2012b). Deterministic criteria for the absence of arbitrage in one-dimensional diffusion models. *Financ. Stoch.* **16**, 225–247.
- [50] NILSEN, W. AND SAYIT, H. (2011). No arbitrage in markets with bounces and sinks. *Int. Rev. Appl. Finan. Issues Econ.* **3**, 696–699.
- [51] PLATEN, E. (2002). Arbitrage in continuous complete markets. *Adv. Appl. Prob.* **34**, 540–558.
- [52] PLATEN, E. (2006). A benchmark approach to finance. *Math. Financ.* **16**, 131–151.
- [53] PLATEN, E. (2011). A benchmark approach to investing and pricing. In *The Kelly Capital Growth Investment Criterion*, eds. L.C. MacLean, E.O. Thorp and W.T. Ziemba, 409–427. World Scientific, Singapore.
- [54] PLATEN, E. AND HEATH, D. (2006). *A Benchmark Approach to Quantitative Finance*. Springer, Berlin - Heidelberg.
- [55] PROTTER, P. (2005). *Stochastic Integration and Differential Equations*, 2nd edn. (v. 2.1). Springer, Berlin - Heidelberg - New York.
- [56] PROTTER, P. AND SHIMBO, K. (2008). No arbitrage and general semimartingales. In *Markov Processes and Related Topics: A Festschrift for Thomas G. Kurtz*, eds. S.N. Ethier, J. Feng and R.H. Stockbridge, 267–283. Institute of mathematical statistics, Beachwood (OH).
- [57] REVUZ, D. AND YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd edn. Springer, Berlin - Heidelberg.
- [58] RHEINLÄNDER, T. AND SCHWEIZER, M. (1997). On L^2 -projections on a space of stochastic integrals. *Ann. Prob.* **25**, 1810–1831.
- [59] RUF, J. (2012). Hedging under arbitrage. To appear in: *Math. Financ.*
- [60] SCHACHERMAYER, W. (2010). The fundamental theorem of asset pricing. In *Encyclopedia of Quantitative Finance*, ed. R. Cont, 792–801. Wiley, Chichester.
- [61] SCHWEIZER, M. (1992). Martingale densities for general asset prices. *J. Math. Econ.* **21**, 363–378.
- [62] SCHWEIZER, M. (1995). On the minimal martingale measure and the Föllmer-Schweizer decomposition. *Stoch. Anal. Appl.* **13**, 573–599.
- [63] SHIRYAEV, A.N. AND CHERNY, A. (2002). Vector stochastic integral and the fundamental theorems of asset pricing. In *Stochastic Financial Mathematics*, collected papers, Tr. Mat. Inst. Steklova **267**, 12–56. Nauka, Moscow.
- [64] STRASSER, E. (2005). Characterization of arbitrage-free markets. *Ann. Appl. Prob.* **15**, 116–124.
- [65] STRICKER, C. AND YAN, J.-A. (1998). Some remarks on the optional decomposition theorem. *Séminaire de Probabilités XXXII*, 56–66, Lecture Notes in Mathematics vol. 1686, Springer.
- [66] TAKAOKA, K. (2010). On the condition of no unbounded profit with bounded risk. To appear in: *Financ. Stoch.*